RELATIONS AND FUNCTIONS 1

The roots of education are bitter but the fruit is sweet.

– Gauss

Mathematicians do not study objects but relations between them. Thus they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevent. They are interested in form only.

– Henri Poincare

1.1 Relations :

Last year we have studied the concept of a relation and a function. We also studied algebraic operations on functions and graphs of relations and functions. We will develop these concepts further in this chapter.

The word 'relation' is used in the context of social obligations also. We will relate the concept of the word 'relation' as used in social and family terms with the word relation as used in mathematics.

We define a relation of the set of human beings H as

$S = \{(x, y) \mid x \in H, y \in H, x \text{ is a brother of } y.\}$

Dev is a brother of Rucha. So ordered pair (Dev, Rucha) $\in$ S.

Let C be the set of all captains of Indian cricket team till 2011.

Let $S = \{(x, y) \mid x \text{ precedes } y, x, y \in C\}$

Then (Kapildev, M. S. Dhoni) $\in$ S.

But (M. S. Dhoni, Kapildev) $\notin$ S.

In the set of natural numbers N, x precedes y, if $y = x + k$ for some $k \in N$. Let $S = \{(x, y) \mid x \text{ precedes } y, x \in N, y \in N\}$. Then $(3, 5) \in S$ as $5 = 3 + 2$. But $(5, 3) \notin S$.

If S is a relation in A i.e. $S \subset (A \times A)$ and $(x, y) \in S$, we say x is related to y by S or xSy.

Let S be a relation in N defined as follows :

$S = \{(x, y) \mid |x - y| \text{ is an even positive integer } x, y \in N\}$, then whenever $(x, y) \in S$, $(y, x) \in S$. **(Why ?)**

Also note that $(x, x) \notin S$.

Now we will define various types of relations.

Void or Empty relation : A relation in the set A with no elements is called an empty relation. $\emptyset \subset (A \times A)$. $\emptyset$ is a relation called empty relation.

The relation S in N defined by

$S = \{(x, y) \mid x + y = 0, x \in N, y \in N\}$ is an empty relation as sum of two positive integers can never be zero.

Universal Relation : A relation in the set A which is $A \times A$ itself is called a universal relation.

The relation S in R defined by

$S = \{(x, y) \mid x \le y \text{ or } y \le x\}$ is universal relation because of the law of trichotomy.

A relation is defined on the set of all living human beings by

S = {(x, y) | Difference between ages of x and y is less than 200 years}. Obviously S is the universal relation.

Reflexive Relation : If S is a relation in the set A and aSa, $\forall a \in A$ i.e. $(a, a) \in S$, $\forall a \in A$, we say S is a reflexive relation.

For example similarity of triangles, congruence of triangles, equality of numbers, subsets in a power set ($A \subset A$ for all $A \in P(U)$) are examples of reflexive relations.

$<$ is not a reflexive relation in R. Infact $a < a$ is false for all $a \in R$.

But $\le$ is reflexive relation on R. $a \le a$, $\forall a \in R$.

Symmetric Relation : If S is a relation in a set A and if $aSb \Rightarrow bSa$ i.e. $(a, b) \in S \Rightarrow (b, a) \in S$ $\forall a, b \in A$, we say S is a symmetric relation in A.

If ABC $\leftrightarrow$ PQR is a similarity relation in the set of triangles in a plane, then PQR $\leftrightarrow$ ABC is a similarity.

In the set of all non-zero integers, we define relation S by $(a, b) \in S \Leftrightarrow d$ divides $a - b$ where d is a fixed non-zero integer.

If m divides $a - b$, then m divides $b - a$. $(a, b) \in S \Rightarrow (b, a) \in S$. If $\Delta PQR \cong \Delta ABC$ then $\Delta ABC \cong \Delta PQR$. These are examples of symmetric relations.

For unequal sets A and B, $A \subset B$ does not imply $B \subset A$.

So $\subset$ is not a symmetric relation in P(U).

Transitive relation : If S is a relation in the set A and if aSb and $bSc \Rightarrow aSc$, $\forall a, b, c \in A$ i.e. $(a, b) \in S$ and $(b, c) \in S \Rightarrow (a, c) \in S$, $\forall a, b, c \in A$, then we say that S is a transitive relation in A.

$\subset$ is a transitive relation in P(U) as $A \subset B$ and $B \subset C \Rightarrow A \subset C$. $\forall A, B, C \in P(U)$.

Similarly $<$ is a transitive relation in R, as $a < b$ and $b < c \Rightarrow a < c$ $\forall a, b, c \in R$.

Equivalence Relation : If a relation S in a set A is reflexive, symmetric and transitive, it is called an equivalence relation in A.

If S is an equivalence relation and $(x, y) \in S$ then we will write, $x \sim y$.

For example equality is an equivalence relation in R, congruence of triangle is an equivalence relation on a set of coplaner triangles.

Example 1 : Prove that congruence $\equiv$ is an equivalence relation in Z.

$x \equiv y \pmod{m}$ (Read : x is congruent to y modulo m) $\Leftrightarrow m$ divides $x - y$, $m \in Z - \{0\}$.

Solution : Reflexivity : $a \equiv a \pmod{m}$ as $a - a = 0$ is divisible by any non-zero integer m.

(**Note :** 0 is divisible by any non-zero real number. But no real number is divisible by 0.)

Symmetry : If $a \equiv b \pmod{m}$, then m dividies $a - b$.

Let $a - b = mn$ $\quad n \in Z$

$\therefore$ $b - a = -mn = m(-n)$ $\quad -n \in Z$

$\therefore$ $b \equiv a \pmod{m}$

$\therefore$ If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$

$\therefore$ $\equiv$ is a symmetric relation in Z.

Transitivity : If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $m \mid (a - b)$ and $m \mid (b - c)$.

($m \mid (a - b)$ means m divides $(a - b)$)

$\therefore$ for some $k \in Z$, $t \in Z$ $a - b = mk$ and $b - c = mt$

$\therefore$ $a - b + b - c = mk + mt$

$\therefore$ $a - c = m(k + t)$ $\quad$ **$k + t \in Z$**

$\therefore$ $a \equiv c \pmod{m}$

If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$

$\therefore$ Congruence relation is an equivalence relation in Z.

Example 2 : Prove that similarity is an equivalence relation in the set of all triangles in a plane.

Solution : For any ΔABC, $\Delta ABC \sim \Delta ABC$ for the correspondence $ABC \leftrightarrow ABC$.

If $\Delta ABC \sim \Delta PQR$, then $\Delta PQR \sim \Delta ABC$.

Also, if $\Delta ABC \sim \Delta PQR$ and $\Delta PQR \sim \Delta XYZ$, then $\Delta ABC \sim \Delta XYZ$.

$\therefore$ $\sim$ is an equivalence relation.

(**Note :** Similarly congruence is an equivalence relation in the set of all triangles in plane.)

Example 3 : A = {the set of all lines in plane}

S = $\{(x, y) \mid x = y$ or x is a line parallel to line $y.\}$

Is S an equivalence relation in A ?

Solution : $(l, l) \in S$ as $l = l$. So, S is reflexive. **(given)**

Let $(l, m) \in S$. So $l \parallel m$ or $l = m$.

If $l \parallel m$, then $m \parallel l$ or if $l = m$, then $m = l$.

$\therefore$ If $(l, m) \in S$ then $(m, l) \in S$.

$\therefore$ S is symmetric.

Let $(l, m) \in S$ and $(m, n) \in S$.

If l, m, n are distinct lines, then $l \parallel m$ and $m \parallel n$ and hence $l \parallel n$.

If $l \parallel m$ and $m = n$ or if $l = m$ and $m \parallel n$, then $l \parallel n$.

If $l = m$ and $m = n$, then $l = n$

$\therefore$ If $(l, m) \in S$ and $(m, n) \in S$, then $(l, n) \in S$.

$\therefore$ S is transitive.

So, S is reflexive, symmetric and transitive.

$\therefore$ S is an equivalence relation.

Example 4 : Prove that the relation $S = \{(a, b) \mid |a - b|$ is even.$\}$ is an equivalence relation in the set $A = \{1, 2, 3, 4, 5, 6, 7\}$.

Solution : | odd integer − odd integer | = | even integer − even integer | = an even integer

$\therefore$ S = {(1, 3), (3, 1), (1, 5) (5, 1), (3, 5), (5, 3), (1, 7), (7, 1), (3, 7), (7, 3), (5, 7), (7, 5), (2, 4), (4, 2), (2, 6), (6, 2), (4, 6), (6, 4), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)}

Since $(x, x) \in S$, $\forall x \in A$. S is reflexive.

Let $(x, y) \in S$.

Hence $|x - y|$ is even.

$|x - y| = |y - x|$. So $|y - x|$ is even. Hence $(x, y) \in S \Rightarrow (y, x) \in S$. So S is symmetric.

Let $(x, y) \in S$ and $(y, z) \in S$.

If $|x - y|$ and $|y - z|$ are even, then x and y have same parity (both even or both odd) and y and z have same parity. Thus x and z have same parity.

$\therefore$ $|x - z|$ is even.

$\therefore$ $(x, z) \in S$, if $(x, y) \in S$ and $(y, z) \in S$

$\therefore$ S is transitive.

So, S is reflexive, symmetric and transitive.

$\therefore$ S is an equivalence relation.

Antisymmetric Relation : If S is a relation in A and if $(a, b) \in S$ and $(b, a) \in S \Rightarrow a = b$, $\forall a, b \in A$ then S is said be an antisymmetric relation.

$\subset$ is an antisymmetric relation in the set P(U) as $A \subset B$ and $B \subset A \Rightarrow A = B$, $\forall A, B \in P(U)$

$\leq$ is an antisymmetric relation in R because $a \leq b$ and $b \leq a \Rightarrow a = b$ $\forall a, b \in R$

Example 5 : Give an example of a relation which is (1) reflexive and symmetric but not transitive (2) reflexive and transitive but not symmetric (3) symmetric and transitive but not reflexive.

Solution :

(1) A = the set of all lines in plane.

S = $\{(x, y) \mid x = y$ or x is perpendicular to y, $x, y \in$ A$\}$ is a relation in A.

Since $l = l$, $(l, l) \in$ S. So S is reflexive.

If $(l, m) \in$ S, then $l = m$ or l is perpendicular to m.

$\therefore$ $m = l$ or m is perpendicular to l.

$\therefore$ $(m, l) \in$ S.

$\therefore$ $(l, m) \in$ S $\Rightarrow (m, l) \in$ S.

So S is symmetric.

Let $(l, m) \in$ S and $(m, n) \in$ S and $l \neq m$, $m \neq n$, $l \neq n$

Hence $l \perp m$ and $m \perp n$. So $l \parallel n$, as $l \neq n$.

$\therefore$ $(l, n) \notin$ S

$\therefore$ S is reflexive and symmetric but not transitive.

(2) $\leq$ in R is reflexive and transitive but not symmetric.

$a \leq a \quad \forall a \in$ R. So, S is reflexive.

$a \leq b$ and $b \leq c \Rightarrow a \leq c \quad \forall a, b\ c \in$ R. So S is transitive.

but if $a \leq b$, then $b \nleq a$, unless $a = b$.

$\therefore$ S is not symmetric.

Thus $(3, 5) \in$ S, but $(5, 3) \notin$ S where S is the relation $\leq$.

$\therefore$ S is reflexive and transitive but not symmetric.

(3) Let A = $\{1, 2, 3\}$.

S = $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$

S is symmetric and transitive but not reflexive as $(3, 3) \notin$ S

Example 6 : Give an example of a relation which is **(1)** reflexive but not symmetric or transitive **(2)** symmetric but not reflexive or transitive **(3)** transitive but not reflexive or symmetric.

Solution : (1) Let A = $\{1, 2, 3\}$.

S = $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$

$(1, 1), (2, 2), (3, 3)$ are in S. Hence S is reflexive.

$(1, 2) \in$ S but $(2, 1) \notin$ S. Hence S is not symmetric.

$(1, 2) \in$ S, $(2, 3) \in$ S but $(1, 3) \notin$ S.

$\therefore$ S is not transitive.

$\therefore$ S is reflexive but neither symmetric nor transitive.

(2) Let A = $\{1, 2, 3\}$, S = $\{(1, 2), (2, 1)\}$

S is symmetric but neither reflexive nor transitive.

(3) Consider $<$ in the set R.

$a < b$ and $b < c \Rightarrow a < c \quad \forall a, b\ c \in$ R. So, S is transitive.

but $a \nless a$ and if $a < b$ then $b \nless a$. So, S is neither reflexive nor symmetric.

$\therefore$ $<$ is transitive but neither reflexive nor symmetric.

Example 7 : Give an example of a relation which is not reflexive, not symmetric, not transitive.

Solution : Let A = $\{1, 2, 3\}$, S = $\{(1, 1), (2, 2), (1, 2), (2, 3)\}$.

$(3, 3) \notin$ S. So S is not reflexive.

$(1, 2) \in$ S but $(2, 1) \notin$ S. So S is not symmetric.

$(1, 2) \in$ S and $(2, 3) \in$ S but $(1, 3) \notin$ S. So S is not transitive.

$\therefore$ S is not reflexive, not symmetric, not transitive.

Example 8 : Following is a proof that a relation which is symmetric and transitive is also reflexive. Find what is wrong with it.

Let $x\text{S}y$

$\therefore$ $y\text{S}x$ **(Symmetry)**

Since $x\text{S}y$ and $y\text{S}x$, so $x\text{S}x$ **(Transitivity)**

$\therefore$ S is reflexive.

Solution : This is not correct argument.

There may be some x such that $x\text{S}y$ is not true for any y in set A.

Then the argument fails.

For example let A = {1, 2, 3, 4}

S = {(1, 1), (2, 2), (1, 2), (2, 1), (1, 3), (3, 1), (3, 3), (2, 3), (3, 2)}

(4, 4) $\notin$ S. This is because for no x, $(x, 4) \in$ S.

$\therefore$ S is not reflexive even though it is symmetric and transitive..

Example 9 : A relation S is said to be circular if $x\text{S}y$ and $y\text{S}z$ implies $z\text{S}x$. Prove that if a relation is reflexive and circular, it is an equivalence relation.

Solution : S is reflexive. **(given)**

Let $x\text{S}y$. We already have $y\text{S}y$.

$\therefore$ $x\text{S}y$ and $y\text{S}y \Rightarrow y\text{S}x$

$\therefore$ $x\text{S}y \Rightarrow y\text{S}x$

$\therefore$ S is symmetric.

Let $x\text{S}y$ and $y\text{S}z$.

$\therefore$ $z\text{S}x$ **(S is circular)**

$\therefore$ $x\text{S}z$ **(S is symmetric)**

$\therefore$ S is transitive.

$\therefore$ S is an equivalence relation.

Arbitrary Union : Let I be a non-empty set of real numbers. Let A_i be a set corresponding to $i \in$ I

Then we define $\bigcup\limits_{i \in I} A_i = \{x \mid x \in A_i \text{ for at least one } i \in I\}$

$$\bigcap\limits_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

For example, let I = [0, 1]. Let $A_i = [0, i]$

Then $\bigcup\limits_{i \in I} A_i = [0, 1]$

$$\bigcap\limits_{i \in I} A_i = \{0\}$$

Equivalence Classes : Let S be an equivalence relation in a set A. If $x\text{S}y$, we say $x \sim y$ (x is equivalent to y) (Read $\sim$ as wiggle)

Let $A_p = \{x \mid x \sim p, x \in A\}$

Let us prove the following :

if $p \sim q$, $A_p = A_q$ and if p is not equivalent to q, $A_p \cap A_q = \emptyset$

If $A_p \cap A_q \neq \emptyset$, let $x \in (A_p \cap A_q)$

$\therefore \quad x \in A_p$ and $x \in A_q$

$\therefore \quad x \sim p$ and $x \sim q$

$\therefore \quad p \sim x$ and $x \sim q$

$\therefore \quad p \sim q$

$\therefore \quad p \in A_q$ and $q \in A_p$

$\therefore \quad A_p \subset A_q$ and $A_q \subset A_p$.

$\therefore \quad A_p = A_q$

Now, if $A_p \cap A_q \neq \emptyset$, then $A_p = A_q$

Also, $p \sim p$.

$\therefore \quad p \in A_p \quad \forall p \in A$.

$$\bigcup_{p \in A} A_p = A$$

Thus an equivalence relation 'partitions' A into disjoint sets A_p such that

(i) $A_p \cap A_q = \emptyset$, if p is not equivalent to q.

(ii) $\bigcup_{p \in A} A_p = A$

These sets A_p are called equivalence classes corresponding to the equivalence relation $\sim$. Conversely any partition of A gives rise to an equivalence relation in A.

We define $x \sim y$ if x and y are in the same class A_p.

$x \sim x$ as x and x belong to the same classes A_p.

If $x \sim y$, then $y \sim x$ because if x and y belong to the same class, then y and x also belong to the same class.

If $x \sim y$ and $y \sim z$, then x and y, y and z belong to the same class. Hence x and z belong to same class.

Hence $x \sim z$

$\therefore \quad \sim$ is an equivalence relation.

Example 10 : We define $a \equiv b \pmod 2$, if $a - b$ is even. Prove $\equiv$ is an equivalence relation in Z. Find equivalence classes.

Solution : $a \equiv a$ as 2 divides 0, or 0 is even.

If $a \equiv b$, then $b \equiv a$ as $a - b$ is even $\Leftrightarrow$ $b - a$ is even.

If $a \equiv b$ and $b \equiv c$, then $a \equiv c$ since $a - b$ is even and $b - c$ is even implies

$a - c = a - b + b - c$ is even.

$\therefore \quad \equiv$ is an equivalence relation.

1, 3, 5,... $\in A_1$ say. ($1 \equiv 3$, $3 \equiv 5$ etc.)

2, 4, 6,... $\in A_2$ say. ($2 \equiv 4$, $4 \equiv 6$ etc.)

All integers are divided into two equivalence classes,

A_1 = the set of odd integers and A_2 = the set of all even integers.

Example 11 : Let $Z = A_1 \cup A_2 \cup A_3$ where $A_1 = \{....1, 4, 7,...\}$
$A_2 = \{....2, 5, 8,...\}$
$A_3 = \{....3, 6, 9,...\}$

Define an equivalence relation whose equivalence classes are A_1, A_2 and A_3.

Solution : Let us define aSb if $3 \mid (a - b)$ or $a \equiv b \pmod 3$.

Then $\equiv$ is an equivalence relation as

$a \equiv a$ as 3 divides $a - a = 0$, so aSa

$$a \equiv b \pmod 3 \Rightarrow 3 \mid (a - b)$$
$$\Rightarrow 3 \mid (b - a)$$
$$\Rightarrow b \equiv a \pmod 3$$

$\therefore$ $aSb \Rightarrow bSa$

$3 \mid (a - b)$ and $3 \mid (b - c)$ implies $3 \mid [(a - b) + (b - c)] = a - c$. Hence aSb and $bSc \Rightarrow aSc$. S is an equivalence relation. So we can write $a \sim b$, if aSb. For this equivalence relation,

$A_1 = \{....1, 4, 7, 10,...\}$, $A_2 = \{....2, 5, 8...\}$, $A_3 = \{....3, 6, 9...\}$ are equivalence classes. For this relation, difference $x - y$ is divisible by 3, if x and y belong to the same class.

Example 12 : Let L be the set of all lines in the XY-plane and S be the relation defined in L as $S = \{(L_1, L_2) \mid L_1 = L_2$ or L_1 is parallel to $L_2\}$. Prove S is an equivalence relation and obtain equivalence classes containing (i) X-axis (ii) Y-axis.

Solution : We have seen that S is an equivalence relation.

The equivalence class of lines containing X-axis is the set of lines $y = b$, $b \in R$.

The equivalence class of lines containing Y-axis is the set of lines $x = a$, $a \in R$.

Example 13 : Show that the set $S = \{(P, Q) \mid$ distance of $P(x, y)$ and $Q(x_1, y_1)$ from origin is same. $P, Q \in R^2\}$ is an equivalence relation. What is the equivalence class containing (1, 0) ?

Solution : $d(P, O) = d(P, O)$. So $(P, P) \in S$. So S is reflexive.

If $d(P, O) = d(Q, O) = r$, then $d(Q, O) = d(P, O) = r$. So S is symmetric.

If $d(P, O) = d(Q, O) = r$ and $d(Q, O) = d(R, O) = r$, then $d(P, O) = d(R, O) = r$

$\therefore$ $(P, Q) \in S, (Q, R) \in S \Rightarrow (P, R) \in S$. Hence S is transitive.

$\therefore$ S is an equivalence relation.

$d(A(1, 0), O) = 1$

The equivalence class containing (1, 0) consists of all points at distance 1 from origin i.e. unit circle.

Exercise 1.1

1. Determine which of the following relations is reflexive, symmetric or transitive ?

(1) $A = \{1, 2, 3,...., 10\}$. $S = \{(x, y) \mid y = 2x\}$

(2) $A = N$, $S = \{(x, y) \mid y$ divides $x\}$

(3) $A = \{1, 2, 3, 4, 5, 6\}$, $S = \{(x, y) \mid y$ divides $x\}$

(4) $A = Z$, $S = \{(x, y) \mid x - y \in Z\}$

(5) $A = R$, $S = \{(x, y) \mid y = x + 1\}$

2. aSb if $6 \mid (a - b)$, $a, b \in Z$. Prove that S is an equivalence relation and write down equivalence classes.

3. Prove $\subset$ is reflexive, antisymmetric and transitive in P(U).

4. (1) $f : N \to N$, $f(x) = x^2$ is a function. We define xSy if $f(x) = f(y)$. Is S an equivalence relation ? What are equivalence classes ?

(2) If $f : Z \to Z$, $f(x) = x^2$, what are equivalence classes for this equivalence relation ?

5. $f : N \times N \to N \times N$, $f((m, n)) = ((n, m))$. We say $(a, b)S(c, d)$ if $f((a, b)) = f((c, d))$. Is S an equivalence relation ? What is the equivalence class containing (1, 2) ?

6. Let L be the set of lines in XY plane. Define a relation S in L by $xSy \Leftrightarrow x = y$ or $x \perp y$ or $x \parallel y$.

   Is S an equivalence relation ? If so, what are equivalence classes ? What is the equivalence class containing X-axis ? What happens if L is the set of all lines in space ?

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1.2 One-one and onto Functions

We have studied the concept of a special type of relation called a function.

Remember, if $A \neq \emptyset$ and $B \neq \emptyset$ and if $f \subset (A \times B)$ and $f \neq \emptyset$ such that for every $x \in A$, there is one and only one $y \in B$ such that $(x, y) \in f$, then f is a function.

Thus f is a relation whose domain is A. We also studied graphs of functions and algebraic operations of addition, subtraction, multiplication and division of functions.

Consider following two functions :

$f : N \to N, \ f(x) = x^2$

$\therefore \ f = \{(1, 1), (2, 4), (3, 9), (4, 16),....\}$

Here $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

$g : Z \to Z \ \ g(x) = x^2$

Then $g = \{(0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4).....\}$

But $-1 \neq 1$ and $g(-1) = g(1) = 1$.

Functions like f are called one-one functions and functions like g are called many-one functions.

Let us give a formal definition.

One-one function : If $f : A \to B$ is a function and if $\forall x_1, x_2 \in A, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, we say $f : A \to B$ is a one-one function, also called an injective function.

Generally we deal with equality with ease rather than working with an inequation. Using contrapositive of defining statement, we can say that if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $\forall x_1, x_2 \in A$, then $f : A \to B$ is a one-one function.

For a function $f : A \to A$, $S = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is an equivalence relation in A.

Obviously $f(x_1) = f(x_1)$ **(Reflexive)**

$f(x_1) = f(x_2) \Rightarrow f(x_2) = f(x_1)$ **(Symmetry)**

$f(x_1) = f(x_2)$ and $f(x_2) = f(x_3) \Rightarrow f(x_1) = f(x_3)$ **(Transitivity)**

$\therefore$ S is an equivalence relation.

For a one-one function $f : A \to A$, the equivalence class containing x_1 is $\{x_1\}$ only.

So $A = \bigcup_{x \in A} \{x\}$. Also $A_i = \{x_i\}$ is the partition of A corresponding to this equivalence relation.

Consider $f : \{1, 2, 3, 4, 5\} \to \{2, 3, 6, 7, 8\}$

$f = \{(1, 2), (2, 2), (3, 3), (4, 6), (5, 6)\}$. f is not a one-one function as $1 \neq 2$ and $f(1) = f(2) = 2$.

Many-one function : If $f : A \to B$ is a function and if $\exists x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, then $f : A \to B$ is said to be a many-one function.

See that this defining statement is the negation of the statement used to define a one-one function.

We define $f(C) = \{y \mid y = f(x), x \in C, C \subset A, C \neq \emptyset\}$ and

$$f^{-1}(D) = \{x \mid y = f(x), x \in A, y \in D, D \subset B\}$$

See that $f(C)$ and $f^{-1}(D)$ are merely symbols.

We note that $f(C)$ is never empty. Set $f^{-1}(D)$ could be $\emptyset$.

In this example if $C = \{2, 3, 4\}$, $f(C) = \{2, 3, 6\}$

If $C = \{1, 2\}$, $f(C) = \{2\}$

If $D = \{8\}$, $f^{-1}(D) = \emptyset$

If $D = \{2\}$, $f^{-1}(D) = \{1, 2\}$

If $D = \{2, 6\}$, $f^{-1}(D) = \{1, 2, 4, 5\}$

In fact $f(A)$ is the range of $f : A \to B$.

$f^{-1}(D)$ is the set of pre-images of the elements of D.

$f^{-1}(B) = A$

Let us see some examples.

Example 14 : Determine whether $f : N \to N$, $f(x) = 2x$ is one-one or not.

Solution : Let $x_1, x_2 \in N$.

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$\therefore$ $f : N \to N$, $f(x) = 2x$ is one-one.

Example 15 : If $f : R \to Z$, $f(x) = [x]$ = integer part of x (or floor function $\lfloor x \rfloor$), is $f : R \to Z$ one-one ?

Solution : No. $f(2.1) = [2.1] = 2$

$f(2.23) = [2.23] = 2$

$\therefore$ $f : R \to Z$, $f(x) = [x]$ is not one-one.

Example 16 : Is $f : R \to R^+ \cup \{0\}$, $f(x) = |x|$ one-one ?

Solution : No. $f(-1) = f(1) = 1$

$\therefore$ $f : R \to R^+ \cup \{0\}$, $f(x) = |x|$ is not one-one.

Example 17 : If $f : N \cup \{0\} \to N \cup \{0\}$, $f(x) = x - 3\left[\frac{x}{3}\right]$, is f one-one ? Find equivalence classes for the relation $S = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$.

Solution : $f(1) = 1 - 3\left[\frac{1}{3}\right] = 1$, $f(2) = 2$, $f(3) = 3 - 3 = 0$, $f(4) = 4 - 3\left[\frac{4}{3}\right] = 1$,

$$f(5) = 5 - 3\left[\frac{5}{3}\right] = 2, f(6) = 6 - 3\left[\frac{6}{3}\right] = 0.$$

In fact $f(n)$ = the remainder when n is divided by 3.

$\therefore$ $f(1) = f(4) = f(7) = f(10) = ... = 1$

$f(2) = f(5) = f(8) = f(11) = ... = 2$

$f(3) = f(6) = f(9) = f(12) = ... = 0$

$\therefore$ $f : N \cup \{0\} \to N \cup \{0\}$, $f(x) = x - 3\left[\frac{x}{3}\right]$ is not one-one.

The equivalence classes are $\{1, 4, 7, 10,...\}$, $\{2, 5, 8, 11,...\}$, $\{0, 3, 6, 9, 12,...\}$

Onto Function : If the range of the function $f : A \rightarrow B$ is B, we say that f is an onto function or surjective function or more precisely f is a function from A onto B.

If $R_f = f(A) = B$, f is onto.

Thus, if there exists at least one $x \in A$ corresponding to every $y \in B$, such that $y = f(x)$, $f : A \rightarrow B$ is an onto function. If $\exists y \in B$, for which there is no $x \in A$ such that $y = f(x)$, $f : A \rightarrow B$ is not an onto function.

Example 18 : Give one example each of a function which is (1) one-one and onto, (2) one-one and not onto, (3) many-one and onto, (4) many-one and not onto.

Solution : (1) $f : N \rightarrow E$, E being the set of even natural numbers, $f(x) = 2x$.

$f = \{(1, 2), (2, 4), (3, 6), ...\}$

$\therefore$ $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

$R_f = \{2, 4, 6, ...\} = E$

Infact every $y \in E$ is of the form $2n$ for some $n \in N$ and $f(n) = 2n = y$

$\therefore$ $R_f = E$

$\therefore$ f is an onto function.

(2) $f : N \rightarrow N$, $f(x) = 2x$

$f = \{(1, 2), (2, 4), (3, 6), ...\}$

f is one-one as in (1).

$\therefore$ $R_f = \{2n \mid n \in N\} = E$, the set of even natural numbers.

$\therefore$ $R_f = E \neq N$

$\therefore$ f is not an onto function.

(3) $f : R \rightarrow Z$, $f(x) = [x]$

$f(1.1) = 1$, $f(1.3) = 1$

$\therefore$ f is many-one.

But $R_f = Z$, since for every $n \in Z$, $f(n) = n$. Thus every integer is in the range of f.

$\therefore$ f is onto.

(4) $f : Z \rightarrow Z$, $f(x) = x^2$

$f(-1) = f(1) = 1$. So f is not one-one, but it is many-one.

$R_f = \{0, 1, 4, 9, ...\} \neq Z$

$\therefore$ f is not onto.

One-one and onto function :

If $f : A \rightarrow B$ is a one-one and onto function, it is called a bijective function.

Example 19 : Prove that $f : R \rightarrow R$, $f(x) = ax + b$ $a \neq 0$ is a bijective function.

Solution : Let $f(x_1) = f(x_2)$

$\therefore$ $ax_1 + b = ax_2 + b$

$\therefore$ $ax_1 = ax_2$

$\therefore$ $x_1 = x_2$ $\qquad (a \neq 0)$

$\therefore$ f is one-one.

$y = ax + b \Leftrightarrow x = \frac{y-b}{a}$ $\qquad (a \neq 0)$

$\therefore$ For every $y \in R$, $\exists x \in R$ such that,

$$f(x) = f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y - b + b = y$$

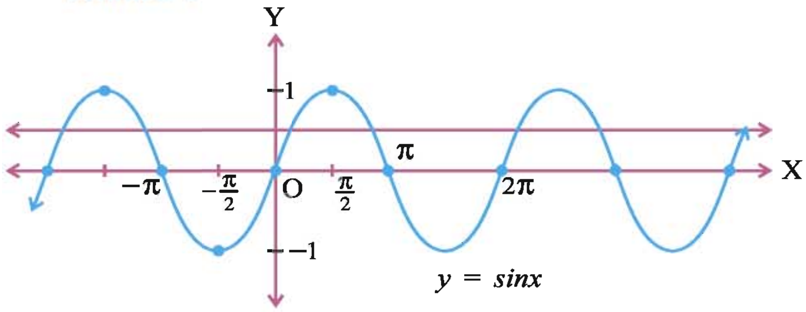
∴ Range of f is R.

∴ $f : R \to R$ is onto.

∴ $f : R \to R$ is a bijective function.

Example 20 : In how many points does a horizontal line intersect the graph of $y = f(x)$, if f is one-one ?

Solution :



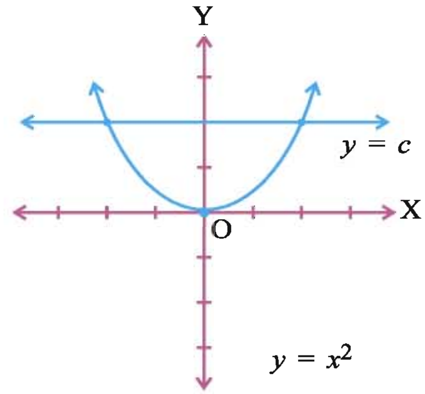


Figure 1.1

The graph of a one-one function $f : A \to B$ is intersected by a horizontal line $y = c$ in at most one point.

For $f : R \to R$, the graph of $f(x) = x^2$ is intersected by a horizontal line $y = c$ in two points in general ($c > 0$). For $x_1 \neq x_2$, we should have $f(x_1) \neq f(x_2)$. So if restrict the function to $f : R^+ \to R^+$, $f(x) = x^2$, it is one-one. The same thing happens in the case of graph of $y = sinx$. If $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\left[0, \frac{\pi}{2}\right]$ etc, the graph of $y = sinx$ is intersected by line $y = c$ $(-1 \leq c \leq 1)$ in at most one point. Otherwise the line $y = c$ intersects the graph of $y = sinx$ in infinitely many points. $(-1 \leq c \leq 1)$

Example 21 : If $A = \{x_1, x_2, x_3, ..., x_n\}$, prove any function $f : A \to A$ is injective if and only if it surjective.

Solution : Let $f : A \to A$ be one-one.

∴ $f(x_1), f(x_2), ..., f(x_n)$ are all distinct elements of A.

But A has n elements $x_1, x_2, ..., x_n$ only.

∴ $f(x_1), f(x_2), ..., f(x_n)$ must be $x_1, x_2, x_3, ..., x_n$ in some order.

∴ $R_f = A$

∴ $f : A \to A$ is onto.

Conversely, suppose $f : A \to A$ is onto.

∴ $R_f = \{x_1, x_2, x_3, ..., x_n\}$

Now, $\{f(x_1), f(x_2), ..., f(x_n)\} = \{x_1, x_2, x_3, ..., x_n\}$.

∴ No $f(x_i)$ can be equal to $f(x_j)$. $(i \neq j)$

(If some $f(x_i) = f(x_j)$, R_f will not contain all $x_1, x_2, x_3, ..., x_n$.)

∴ f is one-one.

Example 22 : If $f : \{x_1, x_2, ..., x_m\} \to \{y_1, y_2, ..., y_n\}$ is one-one, prove that $m \leq n$.

Solution : f is one-one.

∴ $f(x_1), f(x_2), ..., f(x_m)$ are m distinct elements from amongst $\{y_1, y_2, ..., y_n\}$

∴ $m \leq n$

Example 23 : If $f : \{x_1, x_2, ..., x_m\} \to \{y_1, y_2, ..., y_n\}$ is onto, prove that $m \geq n$.

Solution : Some of $f(x_1), f(x_2), ..., f(x_m)$ may be equal but they must form the set $\{y_1, y_2, ..., y_n\}$.

$\therefore$ If $m < n$, at most m elements out of $\{y_1, y_2, ..., y_n\}$ will be in the range, not all $y_1, y_2, ..., y_n$.

$\therefore$ $m \geq n$

(**Note :** If A, B are finite sets and $f : A \to B$ is bijective then $n(A) = n(B)$.

Example 24 : $f : N \to Z$, $f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$

Prove that f is bijective.

Solution : $f = \{(1, 0), (2, 1), (3, -1), (4, 2), ...\}$

as $f(1) = -\frac{1-1}{2} = 0$ (1 odd)

$f(2) = \frac{2}{2} = 1$ (2 even) etc.

$\therefore$ If n is a positive integer, $f(2n) = \frac{2n}{2} = n$. Since $2n \in N$, $2n \in D_f$, $2n$ is even.

If n is a negative integer or zero, $f(-2n + 1) = -\left(\frac{-2n+1-1}{2}\right) = n$.

If n is a negative integer or zero, $-2n + 1 \in N$. $-2n + 1$ is odd.

$\therefore$ All integers are in the range of given $f : N \to Z$.

$\therefore$ $R_f = Z$. So f is surjective.

$f(n) = \frac{n}{2}$ or $-\left(\frac{n-1}{2}\right)$

$\frac{n_1}{2} = \frac{n_2}{2} \Rightarrow n_1 = n_2$, $-\frac{n_1-1}{2} = -\frac{n_2-1}{2} \Rightarrow n_1 = n_2$

and $\frac{n_1}{2} = -\frac{n_2-1}{2} \Rightarrow n_1 + n_2 = 1$, impossible.

$\therefore$ $f(n_1) \neq f(n_2)$ for any $n_1, n_2 \in N$.

$\therefore$ f is one-one.

$\therefore$ f is bijective.

Example 25 : Prove that $f : R - \{2\} \to R - \{2\}$, $f(x) = \frac{2x-1}{x-2}$ is one-one and onto.

Solution : $f(x_1) = f(x_2) \Rightarrow \frac{2x_1-1}{x_1-2} = \frac{2x_2-1}{x_2-2}$

$\Rightarrow 2x_1x_2 - x_2 - 4x_1 + 2 = 2x_1x_2 - x_1 - 4x_2 + 2$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Let $x \in R - \{2\}$.

Let $y = f(x) = \frac{2x-1}{x-2}$ where $x = R - \{2\}$

$\therefore$ $xy - 2y = 2x - 1$

$\therefore$ $(y - 2)x = 2y - 1$ $(y \neq 2)$

$\therefore$ $x = \frac{2y-1}{y-2}$

$\therefore$ For every $y \in R - \{2\}$, there is $x \in R - \{2\}$ such that,

$$y = f(x), \text{ since } f(x) = f\left(\frac{2y-1}{y-2}\right) = \frac{2\left(\frac{2y-1}{y-2}\right)-1}{\frac{2y-1}{y-2}-2}$$

$$= \frac{4y-2-y+2}{2y-1-2y+4}$$

$$= y$$

$\therefore$ $R_f = R - \{2\}$

$\therefore$ f is onto.

Example 26 : $f : N \times N \to N$, $f((m, n)) = m + n$. Is f one-one ? Is f onto ?

Solution : $f((1, 2)) = 1 + 2 = 3$, $f((2, 1)) = 2 + 1 = 3$

but $(1, 2) \neq (2, 1)$.

$\therefore$ f is not one-one.

$m \geq 1, n \geq 1 \Rightarrow m + n \geq 2$

$\therefore$ $f((m, n)) \geq 2, \forall (m, n) \in N \times N$

$\therefore$ $1 \notin R_f$

$\therefore$ f is not onto.

Example 27 : $f : N \times N \to N \times N$, $f((m, n)) = (n, m)$. Prove f is bijective.

Solution : $f((m_1, n_1)) = f((m_2, n_2)) \Rightarrow (n_1, m_1) = (n_2, m_2)$

$\Rightarrow n_1 = n_2, m_1 = m_2$

$\Rightarrow (m_1, n_1) = (m_2, n_2)$

$\therefore$ f is one-one.

$\forall (m, n) \in N \times N, f((n, m)) = (m, n)$

$\therefore$ $R_f = N \times N$

$\therefore$ f is onto.

Exercise 1.2

Are following functions one-one ? Are they onto ? (1 to 11)

1. $f : R \to R$, $f(x) = 5x + 7$
2. $f : R \to R$, $f(x) = 2 - 3x$
3. $f : R \to R$, $f(x) = x^2 + 4x + 5$
4. $f : R \to R$, $f(x) = x^2 - x - 2$
5. $f : N \to N$, $f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{n+1}{2} & n \text{ is odd} \end{cases}$
6. $f : R \to (-1, 1)$, $f(x) = \frac{x}{1+|x|}$
7. $f : A \times B \to A$, $f((a, b)) = a$, A and B are not singleton, $A \neq \emptyset$, $B \neq \emptyset$.
8. $f : R \to R$, $f(x) = x^3$
9. $f : Z \to Z$, $f(n) = \begin{cases} n + 2 & n \text{ is even} \\ 2n + 1 & n \text{ is odd.} \end{cases}$

10. $f : Z \to Z,\ f(n) = \begin{cases} n+1 & n \text{ even} \\ n-3 & n \text{ odd.} \end{cases}$

11. $f : Z \to Z,\ f(n) = \begin{cases} n-2 & n \text{ even} \\ 2n+2 & n \text{ odd.} \end{cases}$

12. How many one-one functions are there from $\{1, 2, 3, ..., n\}$ to itself ?

13. $A_1 = \{1\}, A_2 = \{1, 2\}, A_3 = \{1, 2, 3\}$

How many onto functions $f : A_i \to A_i$ $(i = 1, 2, 3)$ are there ? Can you generalize the result ?

*

1.3 Composite Functions

We have studied the concept of composite functions. Let us revise it.

If $f : A \to B$ and $g : B \to C$ are two functions, their composite function $gof : A \to C$ is defined by

$$(gof)(x) = g(f(x))$$

If $f : A \to B$ and $g : C \to D$ are functions and $R_f \subset D_g$, $gof : A \to D$ is defined by

$$(gof)(x) = g(f(x))$$

Example 28 : If $f : N \to N,\ f(x) = 2x + 3$ and $g : N \to N,\ g(x) = 5x + 7$, find gof and fog.

Solution : $gof : N \to N$

$$(gof)(x) = g(f(x)) = g(2x + 3) = 5(2x + 3) + 7 = 10x + 22$$

$fog : N \to N$

$$(fog)(x) = f(g(x)) = f(5x + 7) = 2(5x + 7) + 3 = 10x + 17$$

In general, $gof \neq fog$.

Example 29 : If $f : R \to R,\ f(x) = x^3$ and $g : R \to R\ \ g(x) = x^5$, prove that $gof = fog$.

Solution : $gof : R \to R,\ (gof)(x) = g(f(x)) = g(x^3) = (x^3)^5 = x^{15}$

$fog : R \to R,\ (fog)(x) = f(g(x)) = f(x^5) = (x^5)^3 = x^{15}$

Here $fog = gof$

(**Note :** Obviously $(a^m)^n = (a^n)^m = a^{mn}$)

Example 30 : $f : \{1, 2, 4, 5\} \to \{2, 3, 6, 7\}$

$f = \{(1, 2), (2, 3), (4, 6), (5, 7)\}$ and

$g : \{2, 3, 6, 7, 8\} \to \{1, 3, 5, 6\}$

$g = \{(2, 1), (3, 1), (6, 1), (7, 5), (8, 6)\}$. Find gof and fog whichever is possible.

Solution : $R_f = \{2, 3, 6, 7\} \subset D_g = \{2, 3, 6, 7, 8\}$

$\therefore$ gof exists.

$\therefore$ $gof = \{(1, 1), (2, 1), (4, 1), (5, 5)\}$
as $(gof)(1) = g(f(1)) = g(2) = 1$, $(gof)(2) = g(f(2)) = g(3) = 1$ etc.

$R_g = \{1, 5, 6\} \not\subset D_f = \{1, 2, 4, 5\}$

$\therefore$ fog does not exist.

Example 31 : If $f : A \rightarrow B$ and $g : B \rightarrow C$ are one-one functions, prove that $gof : A \rightarrow C$ is one-one.

Solution : $(gof)(x_1) = (gof)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$ **($x_1, x_2 \in A$)**

$\Rightarrow f(x_1) = f(x_2)$ **(g is one-one)**

$\Rightarrow x_1 = x_2$ **(f is one-one)**

$\therefore$ $gof : A \rightarrow C$ is one-one.

Example 32 : If $f : A \rightarrow B$ is onto B and $g : B \rightarrow C$ is onto C, prove that, $gof : A \rightarrow C$ is onto C.

Solution : Let $y \in C$.

Since $g : B \rightarrow C$ is onto C, there exists $z \in B$ such that $g(z) = y$.

Now, $f : A \rightarrow B$ is onto B and $z \in B$.

$\therefore$ $\exists x \in A$ such that $f(x) = z$

$\therefore$ $g(z) = y \Rightarrow g(f(x)) = y$

$\therefore$ $(gof)(x) = y$

$\therefore$ For every $y \in C$, $\exists x \in A$ such that $(gof)(x) = y$

$\therefore$ $gof : A \rightarrow C$ is onto C.

Example 33 : If $gof : A \rightarrow C$ is one-one, can you say $f : A \rightarrow B$ and $g : B \rightarrow C$ are one-one ?

Solution : No.

Let $f : A \rightarrow B$, $A = \{1, 2, 3, 4, 5\}$, $B = \{5, 6, 7, 8, 9, 10, 11\}$

$f = \{(1, 5), (2, 6), (3, 7), (4, 8), (5, 9)\}$

Let $g : B \rightarrow B$, $g(x) = x + 1$, if $x \neq 10$ or 11

$g(10) = g(11) = 5$

Then $gof : A \rightarrow B$, $gof = \{(1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$ is one-one.

But $g : B \rightarrow B$ is not one-one.

[**Note :** Here we have taken B = C.]

Example 34 : If $f : A \rightarrow B$ and $g : B \rightarrow C$ are two functions and $gof : A \rightarrow C$ is one-one, then prove that $f : A \rightarrow B$ is one-one.

Solution : Let $f(x_1) = f(x_2)$ **$x_1, x_2 \in A$**

$\therefore$ $g(f(x_1)) = g(f(x_2))$ **($f(x_1) \in B$, $f(x_2) \in B$)**

$\therefore$ $(gof)(x_1) = (gof)(x_2)$

$\therefore$ $x_1 = x_2$ **(gof is one-one)**

$\therefore$ $f : A \rightarrow B$ is one-one.

Example 35 : If $gof : A \rightarrow C$ is onto C, are $f : A \rightarrow B$ and $g : B \rightarrow C$ onto C ?

Solution : No. Let $f : \{1, 2, 3, 4\} \rightarrow \{2, 3, 4, 5, 6, 7\}$, $f(x) = x + 1$

$g : \{2, 3, 4, 5, 6, 7\} \rightarrow \{4, 6, 8, 10\}$, $g(x) = 2x$ if $x \neq 6$ or 7

$g(6) = g(7) = 10$

Then $gof : \{1, 2, 3, 4\} \to \{4, 6, 8, 10\}$,

$gof = \{(1, 4), (2, 6), (3, 8), (4, 10)\}$

$\therefore$ gof is onto C. But $f : A \to B$ is not onto as $6, 7 \notin R_f$

Example 36 : If $f : A \to B$ and $g : B \to C$ are two functions and if $gof : A \to C$ is onto C, prove that g is onto C.

Solution : $gof : A \to C$ is onto C.

Let $z \in C$

$\therefore$ $\exists x \in A$ such that $(gof)(x) = z$

$\therefore$ $g(f(x)) = z$

$x \in A$ and $f : A \to B$ is a function.

$\therefore$ $f(x) \in B$. Let $y = f(x)$.

$\therefore$ $g(y) = z$, where $y \in B$.

$\therefore$ For every $z \in C$. $\exists y \in B$ such that $g(y) = z$

$\therefore$ $g : B \to C$ is onto C.

Exercise 1.3

1. $f : R \to R$, $g : R \to R$, $h : R \to R$ are functions.

Prove : (i) $(fog)oh = fo(goh)$ (2) $(f + g)oh = foh + goh$

2. Find gof and fog for

(1) $f : R \to R$, $f(x) = |x|$, $g : R \to R$, $g(x) = x^2$

(2) $f : R^+ \to R^+$, $f(x) = x^3$, $g : R^+ \to R^+$, $g(x) = x^{\frac{1}{3}}$

3. $f : R^+ \to R$, $f(x) =$ cube root of $(3 - x^3)$. Find fof.

4. $f : R \to R$, $f(x) = x^2 - x - 2$. Find fof.

5. $f : R - \{-1\} \to R - \{-1\}$, $f(x) = \frac{1-x}{1+x}$. Find fof.

6. $f : R \to R$ is signum function.

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$g : R \to Z$, $g(x) = [x]$. Find fog and gof.

7. $f : Z \to Z$ and $g : Z \to Z$ are defined as follows :

$$f(n) = \begin{cases} n + 2 & n \text{ even} \\ 2n - 1 & n \text{ odd} \end{cases} \qquad g(n) = \begin{cases} 2n & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

Find fog and gof.

8. (1) If $A \neq \emptyset$, $B \neq \emptyset$ and $f : A \to B$ is a one-one function, prove that there exists a function $g : B \to A$ such that $gof = I_A$. (I is identity function) (g is called left inverse of f.)

(2) If $A \neq \emptyset$, $B \neq \emptyset$ and $f : A \rightarrow B$ is a function onto B, prove that $\exists$ a function $g : B \rightarrow A$ such that $fog = I_B$. (g is called right inverse of f.)

(3) Combine results (1) and (2) if $f : A \rightarrow B$ is a bijective function.

*

1.4 Inverse of a Function

We have $3 \cdot 1 = 3$ as 1 is multiplicative identity. $3 \cdot \frac{1}{3} = 1$ and so $\frac{1}{3}$ is multiplicative inverse of 3. Similarly we have seen in XIth standard that for a function $f : A \rightarrow B$, $foI_A = f$ and $I_Bof = f$ where I_A and I_B are identity functions on A and B respectively. So does there exist a function $g : B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$? The answer is yes under some conditions. We define inverse of a function.

Definition : If $f : A \rightarrow B$ is a function and if there exists a function $g : B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$ we say $g : B \rightarrow A$ is the inverse function of $f : A \rightarrow B$ and denote g by f^{-1}.

The question arises why 'the' inverse ? **We must prove that $g : B \rightarrow A$ is unique before we call it the inverse of $f : A \rightarrow B$ and assign a symbol f^{-1}.**

Uniqueness : Suppose $g : B \rightarrow A$ and $h : B \rightarrow A$ are two inverses of $f : A \rightarrow B$.

$\therefore$ $gof = I_A$, $fog = I_B$, $hof = I_A$, $foh = I_B$.

$g = goI_B = go(foh) = (gof)oh = I_Aoh = h$

Also $g : B \rightarrow A$, $h : B \rightarrow A$ are functions.

$\therefore$ Inverse of a function $f : B \rightarrow A$, if it exists, is unique.

When does the inverse of a function exist ? This is reflected in the following theorems.

Theorem 1.1 : If $f : A \rightarrow B$ has inverse $g : B \rightarrow A$, then $f : A \rightarrow B$ is one-one and onto.

Proof : For $x_1, x_2 \in A$. let $f(x_1) = f(x_2)$

$\therefore$ $g(f(x_1)) = g(f(x_2))$ $\quad$ **($f(x_1), f(x_2) \in B$)**

$\therefore$ $(gof)(x_1) = (gof)(x_2)$

$\therefore$ $I_A(x_1) = I_A(x_2)$ $\quad$ **($g : B \rightarrow A$ is the inverse of $f : A \rightarrow B$)**

$\therefore$ $x_1 = x_2$

$\therefore$ $f : A \rightarrow B$ is one-one.

Let $y \in B$

$\therefore$ $I_B(y) = y$

$\therefore$ $(fog)(y) = y$ $\quad$ **($fog = I_B$)**

$\therefore$ $f(g(y)) = y$

$g : B \rightarrow A$ is a function. $y \in B$. Hence $g(y) \in A$.

Let $g(y) = x$. So $f(g(y)) = f(x) = y$

$\therefore$ $x \in A$ and $f(x) = y$

$\therefore$ For every $y \in B$, there exists $x \in A$ such that $y = f(x)$.

$\therefore$ $f : A \rightarrow B$ is onto B.

Theorem 1.2 : If $f : A \rightarrow B$ is one-one and onto, it has an inverse $g : B \rightarrow A$.

Proof : Let $f(x) = y$ $\quad$ $x \in A$

Define $g(y) = x$

Since $f : A \to B$ is onto, for every $y \in B$ there exists $x \in A$ such that $f(x) = y$ and this x is unique as $f : A \to B$ is one-one.

$\therefore$ $g : B \to A$ is a function.

$(gof)(x) = g(f(x)) = g(y) = x$

$(fog)(y) = f(g(y)) = f(x) = y$

$\therefore$ $gof = I_A$ and $fog = I_B$.

$\therefore$ g is the inverse of f.

A result :

If $f : A \to B$ and $g : B \to C$ are one-one and onto, $gof : A \to C$ is one-one and onto and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof : We know $gof : A \to C$ is one-one and onto. (Ex. 31, 32)

$(gof)^{-1} : C \to A$ exists and $(gof)^{-1} : C \to A$ is a function.

$f^{-1} : B \to A$ and $g^{-1} : C \to B$ are functions.

$\therefore$ $f^{-1}og^{-1} : C \to A$ is a function.

$$(gof)\, o\, (f^{-1}og^{-1}) = go((fof^{-1})og^{-1}) = go(I_B og^{-1}) = gog^{-1} = I_C$$

$$(f^{-1}og^{-1})\, o\, (gof) = f^{-1}o((g^{-1}og)\, of) = f^{-1}o(I_B of) = f^{-1}of = I_A$$

$\therefore$ $(gof)^{-1} = f^{-1}og^{-1}$

Example 37 : For $f : N \to E$, $f(x) = 2x$, find f^{-1} and verify $fof^{-1} = I_E$, $f^{-1}of = I_N$ where E is the set of even natural numbers.

Solution : $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ $f : N \to E$ is one-one.

if $y \in E$, $y = 2n$ For some n, $n \in N$

$f(n) = 2n = y$

$\therefore$ For every $y \in E$, $\exists n \in N$ such that $f(n) = y$

$\therefore$ $f : N \to E$ is onto.

$y = f(x) = 2x \Rightarrow x = \frac{y}{2} \Rightarrow f^{-1}(y) = \frac{y}{2}$ $\quad (x = f^{-1}(y))$

$\therefore$ $f^{-1} : E \to N$, $f^{-1}(y) = \frac{y}{2}$ or $f^{-1}(x) = \frac{x}{2}$

Verification is left to the reader.

Example 38 : $f : R \to R$, $f(x) = ax + b$ $\; a \neq 0$. Find the inverse of $f : R \to R$.

Solution : $f(x_1) = f(x_2) \Rightarrow ax_1 + b = ax_2 + b$

$\Rightarrow ax_1 = ax_2$

$\Rightarrow x_1 = x_2$ $\quad (a \neq 0)$

$\therefore$ f is one-one.

Let $y \in$ R.

$y = ax + b \Rightarrow x = \frac{y-b}{a} \in$ R $\qquad (a \neq 0)$

$\therefore$ For every $y \in$ R, $\exists x \in$ R such that $f(x) = f\left(\left(\frac{y-b}{a}\right)\right) = a\left(\frac{y-b}{a}\right) + b = y$

$\therefore$ f is onto R.

$\therefore$ f^{-1} : R $\rightarrow$ R, $\quad x = f^{-1}(y) = \frac{y-b}{a}$

or we may write f^{-1} : R $\rightarrow$ R, $f^{-1}(x) = \frac{x-b}{a}$

Example 39 : If $f : R^+ \rightarrow R^+$, $f(x) = x^2$, find f^{-1}.

Solution : $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$

$\Rightarrow |x_1| = |x_2|$

$\Rightarrow x_1 = x_2 \qquad (x_1, x_2 \in R^+)$

$\therefore$ f is one-one.

Let $y \in R^+$

$\therefore$ $\exists x \in R^+$ such that $x = \sqrt{y}$ so that $f(x) = x^2 = y$.

$\therefore$ For every $y \in R^+$, $\exists x \in R^+$ such that $f(x) = y$.

$\therefore$ f is onto R^+.

$\therefore$ $f^{-1} : R^+ \rightarrow R^+$, $f^{-1}(y) = \sqrt{y}$

or we may write $f^{-1} : R^+ \rightarrow R^+$, $f^{-1}(x) = \sqrt{x}$

Example 40 : $f : R - \left\{-\frac{3}{2}\right\} \rightarrow R - \left\{\frac{3}{2}\right\}$, $\quad f(x) = \frac{3x+2}{2x+3}$. Find f^{-1}.

Solution : Let $f(x_1) = f(x_2) \qquad x_1, x_2 \in R - \left\{-\frac{3}{2}\right\}$

$\therefore$ $\frac{3x_1+2}{2x_1+3} = \frac{3x_2+2}{2x_2+3}$

$\therefore$ $6x_1x_2 + 9x_1 + 4x_2 + 6 = 6x_1x_2 + 9x_2 + 4x_1 + 6$

$\therefore$ $5x_1 = 5x_2$

$\therefore$ $x_1 = x_2$

$\therefore$ f is one-one.

Let $x \in R - \left\{-\frac{3}{2}\right\}$ and $y = \frac{3x+2}{2x+3}$

$\therefore$ $2xy + 3y = 3x + 2$

$\therefore$ $(2y - 3)x = 2 - 3y$

$\therefore$ $x = \frac{2-3y}{2y-3} \qquad y \neq \frac{3}{2}$

$\therefore$ For every $y \in R - \left\{\frac{3}{2}\right\}$, there exists $x \in R - \left\{-\frac{3}{2}\right\}$ such that $f(x) = y$.

$\therefore$ f is onto.

$\therefore$ $f^{-1} : R - \left\{\frac{3}{2}\right\} \rightarrow R - \left\{-\frac{3}{2}\right\}$, $f^{-1}(y) = -\frac{3y-2}{2y-3}$ or

$f^{-1} : R - \left\{\frac{3}{2}\right\} \rightarrow R - \left\{-\frac{3}{2}\right\}$, $f^{-1}(x) = -\frac{3x-2}{2x-3}$.

Example 41 : If $f : A \to B$ is one-one and onto. Prove $(f^{-1})^{-1}$ exists and $(f^{-1})^{-1} = f$.

Solution : By definition of inverse if $f^{-1} : B \to A$ has inverse $h : A \to B$, it must satisfy $hof^{-1} = I_B$ and $foh^{-1} = I_A$. But $f : A \to B$ does satisfy these conditions and inverse is unique, if it exists.

$\therefore$ $(f^{-1})^{-1}$ exist and $(f^{-1})^{-1} = f$.

Example 42 : $A = \{1, 2, 3\}$, $B = \{1, 4, 9\}$, $f : A \to B$, $f(x) = x^2$. Find f^{-1} and verify $f^{-1}of = I_A$, $fof^{-1} = I_B$.

Solution : $f = \{(1, 1), (2, 4), (3, 9)\}$

$\therefore$ f is one-one.

$R_f = \{1, 4, 9\} = B$

$\therefore$ f is onto B.

$\therefore$ $f^{-1} : B \to A$, $f^{-1}(x) = \sqrt{x}$. $f^{-1} = \{(1, 1), (4, 2), (9, 3)\}$.

$\therefore$ $fof^{-1} = \{(1, 1), (4, 4), (9, 9)\} = I_B$.

$\therefore$ $f^{-1}of = \{(1, 1), (2, 2), (3, 3)\} = I_A$.

Example 43 : For $f : R \to \{x \mid x \geq 5, x \in R\}$, $f(x) = x^2 + 4x + 9$, find f^{-1} if possible.

Solution : $f(x_1) = f(x_2) \Rightarrow x_1^2 + 4x_1 + 9 = x_2^2 + 4x_2 + 9$

$\Rightarrow x_1^2 - x_2^2 + 4(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 4) = 0$

$\Rightarrow x_1 = x_2$ or $x_1 + x_2 + 4 = 0$

Let $x_1 = 0$, $x_2 = -4$ **(To make $x_1 + x_2 + 4 = 0$)**

Then $f(0) = 9$, $f(-4) = 16 - 16 + 9 = 9$

$\therefore$ f is not one-one.

$\therefore$ f^{-1} does not exist.

Example 44 : If $f : R - \{-1\} \to R - \{-1\}$, $f(x) = \frac{1-x}{1+x}$. Prove that f^{-1} exists and show that $f = f^{-1}$.

Solution : $(fof)(x) = f(f(x))$

$$= f\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}}$$

$$= \frac{1+x-1+x}{1+x+1-x}$$

$$= x$$

$\therefore$ $fof = I_A$, where $A = R - \{-1\}$

$\therefore$ By uniqueness of inverse and the definition of f^{-1}, f^{-1} exists and $f = f^{-1}$.

Note : Examples mark with * are only for information, not for examination.

***Example 45 :** If f, g, h are functions from A to A and if *fog and goh* are bijective, prove that f, g, h are bijective.

Solution : (1) First of all we prove that f, g, h are one-one.

Let $g(x_1) = g(x_2) \quad x_1, x_2 \in A$

$\therefore \quad f(g(x_1)) = f(g(x_2)) \quad g(x_1) \in A, g(x_2) \in A$

$\therefore \quad (fog)(x_1) = (fog)(x_2)$

$\therefore \quad x_1 = x_2$ *(fog is one-one)*

$\therefore \quad g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

$\therefore \quad g : A \to A$ is one-one.

Let $h(x_1) = h(x_2) \quad x_1, x_2 \in A$

$\therefore \quad g(h(x_1)) = g(h(x_2)) \quad h(x_1) \in A, h(x_2) \in A$

$\therefore \quad (goh)(x_1) = (goh)(x_2)$

$\therefore \quad x_1 = x_2$ *(goh is one-one)*

$\therefore \quad h(x_1) = h(x_2) \Rightarrow x_1 = x_2$

$\therefore \quad h : A \to A$ is one-one.

Let $f(x_1) = f(x_2) \quad x_1, x_2 \in A$

Since goh is onto A, $\exists y_1, y_2 \in A$ such that,

$(goh)(y_1) = x_1, (goh)(y_2) = x_2$

$\therefore \quad f((goh)(y_1)) = f((goh)(y_2))$ *($f(x_1) = f(x_2)$)*

$\therefore \quad (fog)(h(y_1)) = (fog)(h(y_2))$

$\therefore \quad h(y_1) = h(y_2)$ *(fog is one-one)*

$\therefore \quad g(h(y_1)) = g(h(y_2)) \quad h(y_1), h(y_2) \in A$

$\therefore \quad (goh)(y_1) = (goh)(y_2)$

$\therefore \quad x_1 = x_2$

$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore \quad f : A \to A$ is one-one.

(2) Now we prove f, g, h are onto A.

Let $y \in A$

Since fog is onto A, $\exists z \in A$ such that

$(fog)(z) = y$

$\therefore \quad f(g(z)) = y$

Let $g(z) = x$. Then $x \in A$. Also $f(x) = y$ and $x \in A$

For every $y \in A$, $\exists x \in A$ such that $f(x) = y$.

$\therefore \quad f$ is onto A.

Similarly, since goh is onto A, $\exists z \in A$ such that

$(goh)(z) = y$

$\therefore \quad g(h(z)) = y$

Let $h(z) = x$. Then $g(x) = y$ where $x \in A$

$\therefore \quad g$ is onto A.

Let $y \in$ A. Now $g(y) \in$ A.

Since goh is onto A, $\exists x \in$ A such that

$(goh)(x) = g(y)$

$\therefore$ $g(h(x)) = g(y)$

But g is one-one.

$\therefore$ $h(x) = y$

$\therefore$ For every $y \in$ A, $\exists x \in$ A such that $h(x) = y$.

$\therefore$ h is onto A.

***Example 46 :** $f : A \rightarrow B$ and $g : B \rightarrow C$ and $h : B \rightarrow C$ are functions.

(1) Prove if f is surjective and $gof = hof$, then $g = h$.

(2) Give an example in which $gof = hof$ but $g \neq h$.

Solution : (1) Let $y \in$ B. f is onto B.

$\therefore$ $\exists x \in$ A such that $f(x) = y$

$\therefore$ $g(f(x)) = g(y)$ $\qquad (f(x) \in B)$

$\therefore$ $h(f(x)) = g(y)$ $\qquad (gof = hof)$

$\therefore$ $h(y) = g(y)$

Since $y \in$ B is arbitrary and $g : B \rightarrow C$ and $h : B \rightarrow C$ are functions, $g = h$.

(2) $f : \{1, 2, 3, 4\} \rightarrow \{5, 6, 7\}$

$f = \{(1, 5), (2, 6), (3, 6), (4, 5)\}$

Let $g : \{5, 6, 7\} \rightarrow \{6, 8\}$, $g = \{(5, 6), (6, 8), (7, 8)\}$

Let $h : \{5, 6, 7\} \rightarrow \{6, 8\}$, $h = \{(5, 6), (6, 8), (7, 6)\}$

$gof = \{(1, 6), (2, 8), (3, 8), (4, 6)\}$

$hof = \{(1, 6), (2, 8), (3, 8), (4, 6)\}$

$\therefore$ $gof = hof$. But $g \neq h$

***Example 47 :** If $f : A \rightarrow B$, $g : A \rightarrow B$ are functions and $h : B \rightarrow C$ is a function.

(1) Prove if $hof = hog$ and h is one-one, then $f = g$.

(2) Give an example where $hof = hog$ but $f \neq g$.

Solution : (1) $hof : A \rightarrow C$ and $hog : A \rightarrow C$ are functions.

$(hof)(x) = (hog)(x)$ for $\forall x \in$ A

$\therefore$ $h(f(x)) = h(g(x))$

$\therefore$ $f(x) = g(x) \quad \forall x \in A$ $\qquad$ (h is one-one)

$\therefore$ $f = g$

(2) $f : \{1, 2, 3\} \rightarrow \{4, 5\}$, $\quad f = \{(1, 4), (2, 4), (3, 4)\}$

$g : \{1, 2, 3\} \rightarrow \{4, 5\}$, $\quad g = \{(1, 5), (2, 5), (3, 5)\}$

$h : \{4, 5\} \rightarrow \{6, 7\}$, $\quad h = \{(4, 6), (5, 6)\}$

$hof = \{(1, 6), (2, 6), (3, 6)\} = hog$, but $f \neq g$.

Exercise 1.4

Find f^{-1} if it exists : (1 to 6)

1. $f : R \to R, \quad f(x) = 2x + 3.$
2. $f : Z \to Z, \quad f(x) = x - 7.$
3. $f : R^+ \to R^+, \quad f(x) = x^3.$
4. $f : \{1, 2, 3, 4, ..., n\} \to \{2, 4, 6, ..., 2n\}, f(n) = 2n.$
5. $f : Z \to Z \times \{0, 1\}, f(n) = \begin{cases} \left(\frac{n}{2}, 0\right) & n \text{ even.} \\ \left(\frac{n-1}{2}, 1\right) & n \text{ odd.} \end{cases}$
6. $f : Z \to N, f(n) = \begin{cases} 4n & n > 0, \quad n \text{ even} \\ 4|n| + 1 & n \le 0, \quad n \text{ even} \\ 4n + 2 & n > 0, \quad n \text{ odd} \\ 4|n| + 3 & n < 0, \quad n \text{ odd} \end{cases}$

   (**Hint :** f is not onto. $3 \notin R_f$)
7. For $f : A \to B$, $\exists$ a function $g : B \to A$ such that $gof = I_A$. Prove f is one-one.
8. For $f : A \to B$, $\exists$ a function $h : B \to A$ such that $foh = I_B$. Prove f is onto B.
9. **Examine if following functions have an inverse. Find inverse, if it exists :**

   (1) $f : R \to R, \quad f(x) = \lfloor x \rfloor$ **(Floor function)**

   (2) $f : R \to R^+ \cup \{0\} \quad f(x) = |x|$

   (3) $f : R \to [0, 1), \quad f(x) = x - [x]$

   (4) $f : R \to Z, \quad f(x) = \lceil x \rceil$ **(Ceiling function)**

   (5) $f : C \to C, \quad f(z) = \bar{z}$ **(C = set of complex numbers)**

   (6) $f : N \times N \to N, \quad f((m, n)) = m + n$

   (7) $f : N \times N \to N \times N, \quad f((m, n)) = (n, m)$

*

1.5 Binary Operations

We know that addition of two natural numbers is a natural number.

i.e. $a \in N, b \in N \Rightarrow a + b \in N$.

Similarly $a - b \in Z$ if $a, b \in Z$

$a \times b \in Z$ if $a, b \in Z$

Thus there is a non-empty set X and an ordered pair of elements (a, b) of $X \times X$ giving a unique element of X obtained by so called 'addition', 'multiplication' etc. These are called binary operations on X.

Binary Operation : Let $A \neq \emptyset$. A function $* : A \times A \to A$ is called a binary operation. Instead of notation like $f((a, b))$ or $*(a, b)$, we use the notation $a * b$ for the image of this function

for (a, b) and call $*$ a binary opration on A. Thus, corresponding to $(a, b) \in A \times A$, a unique element $a * b$ of A can be obtained by $*$.

Thus $+$ is a binary operation on N, Z, Q, R, C.

$\times$ is a binary operation on N, Z, Q, R, C.

$-$ is a binary operation on Z, Q, R, C as $a - b$ does not necessarily belong to N if $a \in N$, $b \in N$.

For example $3 \in N$, $7 \in N$, but $3 - 7 = -4 \notin N$.

Similarly $\div$ is a binary operation on $Q - \{0\}$, $R - \{0\}$, $C - \{0\}$. If $b = 0$, $\frac{a}{b}$ is not defined in Q or in R or in C.

If $a \in N$, $b \in N$, then $\frac{a}{b} \notin N$ unless $b \mid a$.

Hence division is not a binary opertion on N.

Commutative law : If $*$ is a binary operation on set A and if $a * b = b * a$, $\forall a, b \in A$, we say $*$ is a commutative operation.

$+$ is commutative on N.

$-$ is not commutative on Z as $a - b \neq b - a$, $a, b \in Z$.

Associative law : If $*$ is a binary operation on A and if $(a * b) * c = a * (b * c)$ $\forall a, b, c \in A$, we say $*$ is an associative binary operation on A.

What is the need of this law ?

See that $(a + b) + c = a + (b + c)$ i.e. $+$ is associative on R. Hence we can write $a + b + c$ without ambiguity for this expression.

$(a - b) - c \neq a - (b - c) \quad \forall a, b, c \in R$

Hence '$-$' is not associative on R. So we have to specify brackets while using '$-$' for three real numbers.

Identity Element : If $*$ is a binary operation on A and if there exists an element e in A such that $a * e = e * a = a$, $\forall a \in A$, we say e is an identity element for $*$.

$0 + a = a + 0 = a, \quad \forall a \in R$

$1 \cdot a = a \cdot 1 = a, \quad \forall a \in R$

$\therefore$ 0 is the additive identity and 1 is the multiplicative identity in R.

$a - 0 \neq 0 - a$ for $a \in R$ unless $a = 0$.

$\therefore$ '$-$' has no additive identity.

Inverse of an element : If $*$ is a binary operation on A with an identity element e and if corresponding to $a \in A$, there exists an element $a' \in A$ such that $a * a' = a' * a = e$ where e is the identity element for $*$, we say a' is an inverse of a and we denote the inverse a' of a by a^{-1}.

$\therefore \quad a * a^{-1} = a^{-1} * a = e$

In R, every non-zero real number a has an inverse $\frac{1}{a}$ for multiplication.

Every element a has an inverse $-a$ for addition in R.

0 has no inverse for multiplication in R.

Operation Table : If A is a finite set and n(A) is 'small', we can prepare a table as follows :

$*$	a_1	a_2	a_3	a_n
a_1				
a_2				
a_3				
.				
.				
.				
a_n				

$a_i * a_j$ is written at the intersection of the ith row and jth column.

If $*$ is commutative, the table is symmetric about the main diagonal.

Example 48 : $*$ is defined on $N \cup \{0\}$ by $a * b = |a - b|$. Is it a binary operation ?

Solution : Yes. If $a \in N \cup \{0\}$, $b \in N \cup \{0\}$, then $a - b \in Z$ and $|a - b| \in N \cup \{0\}$

$\therefore$ $*$ is a binary opeation.

Example 49 : Determine whether following operations $*$ are commutative or not ? associative or not ?

(1) On $N \cup \{0\}$, $a * b = 2^{ab}$

(2) On R^+, $a * b = \frac{a}{b+1}$

Solution : (1) $a * b = 2^{ab} = 2^{ba} = b * a \quad \forall a, b \in N \cup \{0\}$

$\therefore$ $*$ is commutative.

$(2 * 3) * 4 = 2^6 * 4 = 2^{2^6 \cdot 4} = 2^{256}$

$2 * (3 * 4) = 2 * 2^{12} = 2^{2 \cdot 2^{12}} = 2^{2^{13}}$

$\therefore$ $*$ is not associative.

(2) $a * b = \frac{a}{b+1}$, $b * a = \frac{b}{a+1}$

$$\frac{a}{b+1} = \frac{b}{a+1} \Rightarrow a^2 + a = b^2 + b$$

$$\Rightarrow (a - b)(a + b) + (a - b) = 0$$

$$\Rightarrow (a - b)(a + b + 1) = 0$$

$\therefore$ If $a = b$ or $a + b + 1 = 0$, then $a * b = b * a$.

$2 * 3 = \frac{2}{4} = \frac{1}{2} \quad 3 * 2 = \frac{3}{3} = 1$

$\therefore$ $*$ is not commutative.

$$(2 * 3) * 4 = \frac{2}{4} * 4 = \frac{1}{2} * 4 = \frac{\frac{1}{2}}{4+1} = \frac{1}{10}$$

$$2 * (3 * 4) = 2 * \frac{3}{5} = \frac{2}{\frac{3}{5}+1} = \frac{10}{8} = \frac{5}{4}$$

$\therefore$ $*$ is not associative.

Example 50 : $\wedge : R \times R \to R$ is defined by $\wedge(a, b) = a \wedge b = min(a, b)$.

Prepare the operation table for $\wedge$ for the subset $\{2, 3, 4, 7, 8\}$.

Solution :

$\wedge$	2	3	4	7	8
2	2	2	2	2	2
3	2	3	3	3	3
4	2	3	4	4	4
7	2	3	4	7	7
8	2	3	4	7	8

Example 51 : Define $*$ on $\{2, 4, 6, 8\}$ by $a * b = g.c.d.\ (a, b)$.

Prepare the operation table for $*$. Is $*$ commutative ?

Solution :

g.c.d.	2	4	6	8
2	2	2	2	2
4	2	4	2	4
6	2	2	6	2
8	2	4	2	8

Obviously $g.c.d.\ (a, b) = g.c.d.(b, a)$

$\therefore$ $*$ is commutative.

See that the table is symmetric about dotted diagonal.

Example 52 : $*$ is the binary operation on N defined by $a * b = l.c.m.\ (a, b)$

(1) Find $8 * 10$, $5 * 3$, $12 * 24$.

(2) Is $*$ commutative ?

(3) Is $*$ associative ?

(4) Find the identity for $*$, if it exists.

(5) Find inverse of those elements for which it exists.

Solution : (1) $8 * 10 = l.c.m.\ (8, 10) = 40$

$5 * 3 = l.c.m.\ (5, 3) = 15$

$12 * 24 = l.c.m.\ (12, 24) = 24$

(2) $l.c.m.\ (a, b) = l.c.m.\ (b, a)$

$\therefore$ $*$ is commutative.

(3) $*$ is associative.

(4) $a * e = a, \quad \forall a \in$ N means $l.c.m.\ (a, e) = a, \quad \forall a \in$ N

$\therefore$ $e \mid a \quad \forall a \in$ N. In special case $e \mid 1$. So, $e = 1$

Also, $l.c.m.\ (a, 1) = a$.

$\therefore$ 1 is the identity for $l.c.m.$ operation.

(5) $l.c.m.\ (a, b) \geq a$ and $l.c.m.\ (a, b) \geq b$.

$\therefore$ $l.c.m.\ (a, b) \neq 1$ unless $a = b = 1$. Inverse of 1 only exists and it is 1.

Example 53 : Let X $\neq \emptyset$. Prove that union and intersection are binary operations on P(X). Are they commutative ? Are they associative ? Find the identity and inverse if any for $\cup$ and $\cap$.

Solution : $A \cup B \in P(X)$ and $A \cap B \in P(X)$ if $A, B \in P(X)$.

$\therefore$ $\cup$ and $\cap$ and are binary operations on $P(X)$.

Let $A, B, C \in P(X)$.

$A \cup B = B \cup A \quad A \cap B = B \cap A$

and $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

$\therefore$ $\cup$ and $\cap$ and are associative.

Also $A \cup \emptyset = \emptyset \cup A = A$ for all $A \in P(X)$

$\therefore$ $\emptyset$ is the identity for union.

$A \cap X = X \cap A = A$ for all $A \in P(X)$

$\therefore$ X is the identity for intersection.

$A \cup B = \emptyset \Leftrightarrow A = B = \emptyset$.

$\therefore$ $\emptyset$ is the only element of $P(X)$ having $\emptyset$ as the inverse for union.

$(A \cap B) \subset A$. Hence $A \cap B \neq X$ unless $A = B = X$.

$\therefore$ X is the only element of $P(X)$ having inverse X for intersection.

Example 54 : Define $a * b = a + 2b$ on N. Is $*$ commutative ? Is $*$ associative ? Is there any identity or inverse for any element in N ?

Solution : $2 * 3 = 2 + 6 = 8$

$3 * 2 = 3 + 4 = 7$

$\therefore$ $*$ is not commutative.

$(2 * 3) * 4 = 8 * 4 = 8 + 8 = 16$

$2 * (3 * 4) = 2 * 11 = 2 + 22 = 24$

$\therefore$ $*$ is not associative.

If $a * e = e * a = a$, then $a + 2e = e + 2a = a \quad \forall a \in N$

$\therefore$ $a + 2e = a$

$\therefore$ $e = 0$

But $0 \notin N$.

$\therefore$ $*$ has no identity and therefore there is no question of inverse.

Example 55 : $*$ is defined on Z by $a * b = a + b + 1$. Is $*$ associative ? Find the identity and inverse of any element, if it exists.

Solution : $(a * b) * c = (a + b + 1) * c$

$= a + b + 1 + c + 1 = a + b + c + 2$

$a * (b * c) = a * (b + c + 1) = a + (b + c + 1) + 1 = a + b + c + 2$

$\therefore$ $*$ is associative.

Let $a * e = e * a = a$ for $\forall a \in Z$

$\therefore$ $a + e + 1 = a$

$\therefore$ $e = -1$

Also, $a * (-1) = a + (-1) + 1 = a$. Also $(-1) * a = (-1) + a + 1 = a$.

$\therefore$ -1 is the identity for $*$.

$a * b = a + b + 1 = -1 \Rightarrow b = -2 - a$

Also $a * (-a - 2) = a + (-a - 2) + 1 = -1$

$\therefore$ $-a - 2$ is the inverse of a.

Example 56 : Prove if $*$ is an associative binary operation having identity e and if a has an inverse, the inverse is unique.

Solution : Suppose a has two inverses a' and a''.

$\therefore$ $a * a' = a' * a = e$

$a * a'' = a'' * a = e$

Now $a' = a' * e = a' * (a * a'')$

$= (a' * a) * a''$

$= e * a''$

$= a''$

$\therefore$ The inverse is unique.

Example 57 : Define $*$ on R by $a * b = a + b - (ab)^2$.

(1) Prove $*$ is commutative but not associative.

(2) Find the identity element for $*$.

(3) Prove that 1 has two inverses for $*$.

(4) Prove if $a \in$ R, a has at most two inverses.

(5) Which elements have no inverse ? Which have only one inverse ? Which have two inverses ? Find the unique inverse if there is any.

Solution : (1) $a * b = a + b - (ab)^2 = b + a - (ba)^2 = b * a$

$\therefore$ $*$ is commutative.

$(2 * 3) * (-2) = (2 + 3 - 36) * (-2) = (-31) * (-2)$

$= -31 - 2 - (62)^2$

$= -33 - 3844$

$= -3877$

$2 * (3 * (-2)) = 2 * (3 - 2 - (-6)^2) = 2 * (-35)$

$= 2 + (-35) - 4900$

$= -4933$

$\therefore$ $*$ is not associative.

(2) $a * e = a + e - (ae)^2 = e + a - (ae)^2 = a \Rightarrow e - a^2e^2 = 0 \quad \forall a \in \text{R} \Rightarrow e = 0$

(Take in particular $a = 0$)

$a * 0 = a + 0 - 0 = a = 0 * a$

$\therefore$ 0 is the identity for $*$.

(3) Let $1^{-1} = a$.

$1 * a = 1 + a - a^2 = 0$

$\therefore$ $a^2 - a - 1 = 0$

$\therefore$ $a = \frac{1 \pm \sqrt{5}}{2}$

$\therefore$ $1^{-1} = \frac{\sqrt{5} + 1}{2}$ or $\frac{1 - \sqrt{5}}{2}$

$\therefore$ 1 has two inverses.

(4) Let b be inverse of a, $a \in R$.

$\therefore$ $a * b = 0$

$\therefore$ $a + b - a^2b^2 = 0$ **(0 is identity)**

$\therefore$ $b^2a^2 - b - a = 0$

This is a quadratic equation in b.

$\therefore$ It has at most two real roots as $\Delta = 1 + 4a^3$ and Δ may be positive or negative or zero.

$\therefore$ Every element a can have at most two inverses.

If $4a^3 < -1$ or $a < \left(-\frac{1}{4}\right)^{\frac{1}{3}}$, $\Delta < 0$

$\therefore$ a has no inverse.

If $4a^3 > -1$, a has two inverses.

If $a^3 = \frac{-1}{4}$, a has only one inverse.

$\therefore$ If $a = \sqrt[3]{\frac{-1}{4}}$, a has only one inverse, namely $b = \frac{1 \pm \sqrt{1+4a^3}}{2a^2} = \frac{1}{2a^2}$

$\therefore$ $a * \frac{1}{2a^2} = a + \frac{1}{2a^2} - \left(\frac{1}{2a}\right)^2 = a + \frac{1}{2a^2} - \frac{1}{4a^2} = a + \frac{1}{4a^2} = \frac{4a^3+1}{4a^2} = 0$

$\therefore$ $a = \sqrt[3]{\frac{-1}{4}}$ has only one inverse namely $\frac{1}{2a^2}$.

(**Note :** Here $*$ is not associative. Hence uniqueness of inverse cannot be asserted.)

Miscellaneous Examples :

Example 58 : A relation S is said to be triangular, if xSy and xS$z \Rightarrow y$Sz.

Prove S is an equivalence relation $\Leftrightarrow$ S is reflexive and triangular.

Solution : Suppose S is an equivalence relation.

$\therefore$ S is reflexive.

Let xSy and xSz

$\therefore$ ySx and xSz **(S is symmetric)**

$\therefore$ ySz **(S is transitive)**

$\therefore$ xSy and xS$z \Rightarrow y$Sz

$\therefore$ S is triangular.

Conversely let S be reflexive and triangular.

Let xSy. Also xSx.

$\therefore$ ySx

$\therefore$ xS$y \Rightarrow y$Sx

$\therefore$ S is symmetric.

Let xSy and ySz

$\therefore$ ySx and ySz **(S is symmetric)**

$\therefore$ xSz

$\therefore$ S is transitive.

$\therefore$ S is an equivalence relation.

Example 59 : In R, let xSy if $x - y \in Z$. Prove that S is an equivalence relation. What are equivalence classes ?

Solution : $x - x \in Z$ as $0 \in Z$

$\therefore$ xSx

$\therefore$ S is reflexive.

If $x - y \in Z$, then $y - x \in Z$

$\therefore$ $xSy \Rightarrow ySx$

$\therefore$ S is symmetric.

If $x - y \in Z$ and $y - z \in Z$, then

$x - y + y - z = x - z \in Z$

$\therefore$ If xSy and ySz, then xSz

$\therefore$ S is transitive.

$\therefore$ S is an equivalence relation.

So now we can denote S by $\sim$.

Now $x \sim y \Leftrightarrow x - y$ is an integer.

Like if $x = 7.82$, $y = 2.82$, then $x - y = 5 \in Z$

$\therefore$ $x \sim y$

$x - [x] = 7.82 - 7 = 0.82$

$y - [y] = 5.82 - 5 = 0.82$ must be same, if $x \sim y$.

$x - [x]$ consists of those real numbers whose decimal expressions after decimal point are identical.

$x - [x] = y - [y]$ or equivalently $x - y = [x] - [y]$.

The equivalence class of x consists of those real numbers y for which $x - y = [x] - [y]$

Example 60 : Prove $f : R - \{-2\} \to R - \{1\}$, $f(x) = \frac{x}{x+2}$ is one-one and onto. Find f^{-1}.

Solution : $f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$

$\Rightarrow x_1x_2 + 2x_1 = x_1x_2 + 2x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Let $y \in R - \{1\}$, $x \in R - \{-2\}$

Let $y = \frac{x}{x+2}$

$\therefore$ $xy + 2y = x$

$x(y - 1) = -2y$

$x = \frac{-2y}{y-1} = \frac{2y}{1-y}$ $\quad$ $(y \in R - \{1\})$

$\therefore$ For every $y \in R - \{1\}$, $\exists x \in R - \{-2\}$ such that $y = f(x)$

$\therefore$ $R_f = R - \{1\}$

$\therefore$ f is onto $R - \{1\}$.

$\therefore$ $f^{-1} : R - \{1\} \to R - \{-2\}$, $f^{-1}(x) = \frac{2x}{1-x}$

Example 61 : * is defined on R by $a * b = a + b - ab$. Is there an identity for * ? What is inverse of $a \in R$, if it exists ?

Solution : $a * e = e * a = a, \forall a \in R \Rightarrow a + e - ae = a \qquad \forall a \in R$

$\Rightarrow e - ae = 0 \qquad \forall a \in R$

$\Rightarrow e = 0$ **(Take $a = 0$ in particular)**

Also $a * 0 = 0 * a = a + 0 - 0 = a$

$\therefore$ 0 is the identity for $*$.

Now $a * b = a + b - ab = 0 \Rightarrow (1 - a)b = -a$

$\Rightarrow b = \frac{a}{a-1}$, if $a \neq 1$

If $a \neq 1$, a^{-1} exists and $a^{-1} = \frac{a}{a-1}$

Example 62 : Define relation S on $Z - \{0\} \times Z - \{0\}$ by $(a, b)S(c, d) \Leftrightarrow ad = bc$. Prove that it is an equivalence relation. What about equivalence classes ?

Solution : $(a, b)S(a, b)$ as $ab = ba$

$\therefore$ S is reflexive.

If $(a, b)S(c, d)$, then $ad = bc$

$\therefore$ $cb = da$

$\therefore$ $(c, d)S(a, b)$

$\therefore$ S is symmetric.

Let $(a, b)S(c, d)$ and $(c, d)S(e, f)$

$\therefore$ $ad = bc$ and $cf = de$

$\therefore$ $ade = bce$ and $acf = ade$

$\therefore$ $acf = bce$

$\therefore$ $af = be$, since $c \neq 0$

$\therefore$ $(a, b)S(e, f)$

$\therefore$ S is transitive

$\therefore$ S is an equivalence relation.

In fact $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$.

$\therefore$ $\frac{2}{4} \sim \frac{3}{6} \sim \frac{1}{2} \sim \frac{5}{10} \cdots$

$\therefore$ The equivalance class of fractions (a, b) consists of non-zero rational number $\frac{a}{b}$.

Example 63 : Let $*$ be defined by $a * b = \frac{ab}{10}$ for $a, b \in Q^+$

Find the identity element. Find 4^{-1} and $(4 * 5)^{-1}$.

Solution : $a * b = a \Rightarrow \frac{ab}{10} = a \Rightarrow b = 10$ **(as $a \neq 0$)**

Also $a * 10 = 10 * a = \frac{a \cdot 10}{10} = a$

$\therefore$ 10 is the identity for $*$.

Let $4 * a = 10$

$\therefore$ $\frac{4a}{10} = 10$

$\therefore \quad a = 25$

$\therefore \quad 4^{-1} = 25$ $\left(4 * 25 = \frac{4 \cdot 25}{10} = 10\right)$

$\therefore \quad 4 * 5 = \frac{4 \cdot 5}{10} = 2$

Now $2 * a = 10 \Rightarrow \frac{2a}{10} = 10$

$\Rightarrow a = 50$

$\therefore \quad (4 * 5)^{-1} = 2^{-1} = 50$

Exercise 1

1. Prove that there is only one relation in $\{1, 2, 3\}$ which is reflexive and symmetric but not transitive and which contains $(1, 2)$ and $(1, 3)$.

2. Prove that the number of equivalence relations in $\{1, 2, 3\}$ containing $(1, 2)$ is two.

3. S is defined on R by, $(a, b) \in S \Leftrightarrow 1 + ab > 0 \quad \forall a, b \in R$

Prove S is reflexive and symmetric but not transitive.

(**Hint :** Take $a = \frac{1}{3}$, $b = \frac{-1}{2}$, $c = -8$. $(a, b) \in S$, $(b, c) \in S$ and $(a, c) \notin S$)

4. $A = \{1, 2, 3, ..., 14, 15\}$, $S = \{(x, y) \mid y = 5x, x, y \in A\}$

Determine whether S is reflexive, symmetric or transitive.

5. The relation S is defined on R as follows :

$S = \{(a, b) \mid a \le b^2, a, b \in R\}$

Prove S is not reflexive, not symmetric and not transitive.

6. Let $S \subset (R \times R)$. $S = \{(A, B) \mid d(A, B) < 2\}$. Prove S is not transitive.

7. S is defined on $N \times N$ by

$(a, b) \, S \, (c, d) \Leftrightarrow ad(b + c) = bc(a + d)$. Prove that S is an equivalence relation.

8. Determine whether following functions are injective or not ? surjective or not ?

(1) $f : R \to R, \quad f(x) = \begin{cases} 2x + 1 & x \ge 0 \\ x^2 & x < 0 \end{cases}$

(2) $f : R \to R, \quad f(x) = \begin{cases} -x + 1 & x \ge 0 \\ x^2 & x < 0 \end{cases}$

(3) $f : Z \to Z, \quad f(n) = \begin{cases} n - 1 & n \text{ odd} \\ n & n \text{ even} \end{cases}$

(4) $f : Z \to Z, \quad f(n) = \begin{cases} n & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$

(5) $f : R \times (R - \{0\}) \to R, f((x, y)) = \frac{x}{y}$

(6) $f : Z \to Z, f(n) = \begin{cases} n & n \text{ even} \\ 2n + 3 & n \text{ odd} \end{cases}$ (**Hint :** Is $3 \in R_f$?)

(7) $f : [-1, 1] \to [-1, 1], f(x) = x \, |x|$

(8) $f : N \to N \cup \{0\}, f(n) = n + (-1)^n$

(9) $f : N - \{1\} \to N, f(n) =$ largest prime divisor of n.

(10) $f : R - \{3\} \to R - \{1\}, f(x) = \frac{x-2}{x-3}$

(11) $f : R \to R, f(x) = x - [x]$

9. $f : [0, 1] \to [0, 1], f(x) = \begin{cases} x & x \in Q \\ 1 - x & x \notin Q \end{cases}$

Prove $(fof)(x) = x$.

10. $f : Z \to Z, f(n) = 5n$ and

$g : Z \to Z, g(n) = \begin{cases} \frac{n}{5} & \text{if } 5 \mid n \\ 0 & \text{otherwise.} \end{cases}$ Find gof and fog.

11. $f : R \to R, f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

and $g : R \to R, g(x) = [x]$. Prove $(fog)(x) = (gof)(x) \quad \forall x \in [-1, 0)$

12. If $f : A \to B$ and $g : B \to A$ are two functions such that $gof = I_A$, then prove that f is one-one and g is onto A.

13. Prove for functions $f : A \to B$ and $g : B \to C$

(1) If $gof : A \to C$ is onto C, $g : B \to C$ is onto C.

(2) If $gof : A \to C$ is one-one, $f : A \to B$ is one-one.

(3) If $gof : A \to C$ is onto and $g : B \to C$ is one-one, $f : A \to B$ is onto.

(4) If $gof : A \to C$ is one-one and $f : A \to B$ is onto B, $g : B \to C$ is one-one.

14. $f : R^+ \cup \{0\} \to R^+ \cup \{0\}, f(x) = \sqrt{x}, g : R \to R, g(x) = x^2 - 1$. Find fog or gof whichever exists.

15. If $f : N \cup \{0\} \to N \cup \{0\}, f(n) = \begin{cases} n + 1 & n \text{ even} \\ n - 1 & n \text{ odd.} \end{cases}$ Prove $f = f^{-1}$.

16. $f : R \to (-1, 1), f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$. Find f^{-1}, if it exists.

17. $f : R - \left\{\frac{2}{3}\right\} \to R, f(x) = \frac{4x+3}{6x-4}$. Prove $(fof)(x) = x$. What can you say about f^{-1} ?

18. $*$ is defined on R by $a * b = a + b + ab$. Is $*$ commutative ? Is it associative ?

Answer the same question if $a * b = a - b + ab$.

19. Examine whether following binary operations are commutative or not and associative or not :

(1) $a * b = a^b$ on N

(2) $a * b = g.c.d.\ (a, b)$ on N

(3) $a * b = a - b$ on Q

(4) $a * b = a^2 b$ on Q

(5) $a * b = a + b - 5$ on R

(6) $a * b = \frac{a}{b+1}$ on $R - \{-1\}$

(7) $a * b = \frac{a+b}{2}$ on Q

(8) $a * b = \frac{a-b}{2}$ on Q

(9) $a * b = a + b - 2$ on Z

(10) $a * b = a + 2b - 3$ on Z.

20. Find the identity element for following binary operations and inverse of any element in case it exists (provided identity exists) :

(1) $a * b = a + b + ab$ on $Q - \{-1\}$

(2) $a * b = \frac{ab}{2}$ on $Q - \{0\}$

(3) $a * b = a + b - 2$ on Z

(4) $a * b = a + b - ab$ on $R - \{1\}$

(5) $a * b = \sqrt{|a^2 - b^2|}$ on R

(6) $a * b = 3a + 4b - 2$ on R

(7) $a * b = a + 3b^2$ on Z

(8) $a * b = g.c.d.\ (a, b)$ on N.

(9) $A * B = A \cap B$ on P(X) for a non-empty set X.

(10) $A * B = A \cup B$ on P(X) for a non-empty set X.

Section A (1 mark)

1. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The relation S = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)} on {1, 2, 3, 4, 5} is ...... ☐

(a) symmetric only　　(b) reflexive only

(c) transitive only　　(d) an equivalence relation

(2) If A = {1, 2, 3}, then the number of equivalence relation containing (1, 3) is... ☐

(a) 1　　(b) 2　　(c) 3　　(d) 8

(3) S is defined in Z by $(x, y) \in S \Leftrightarrow |x - y| \leq 1$. S is... ☐

(a) reflexive and transitive but not symmetric.

(b) reflexive and symmetric but not transitive.

(c) symmetric and transitive but not reflexive.

(d) an equivalence relation

(4) If S is defined on $R - \{0\}$ by $(x, y) \in S \Leftrightarrow xy \geq 0$. Then S is... ☐

(a) an equivalence relation　　(b) reflexive only

(c) symmetric only　　(d) transitive only

(5) Which of the following defined on Z is not an equivalence relation... ☐

(a) $(x, y) \in S \Leftrightarrow x \geq y$　　(b) $(x, y) \in S \Leftrightarrow x = y$

(c) $(x, y) \in S \Leftrightarrow x - y$ is a multiple of 3　　(d) $(x, y) \in S$ if $|x - y|$ is even

(6) If $a * b = a^2 + b^2$ on Z, then $(2 * 3) * 4 =$ ......

(a) 13 (b) 16 (c) 185 (d) 13

(7) If $a * b = a^2 + b^2 + ab + 2$ on Z, then $3 * 4 =$ ......

(a) 40 (b) 39 (c) 25 (d) 41

(8) If $a * b = \frac{ab}{2}$ on Q^+, then the identity for $*$ is ......

(a) 2 (b) 3 (c) 0 (d) 1

(9) If $a * b = \frac{ab}{3}$ on Q^+, then the inverse of a ($a \neq 0$) for $*$ is ......

(a) $\frac{3}{a}$ (b) $\frac{9}{a}$ (c) $\frac{1}{a}$ (d) $\frac{2}{a}$

(10) The number of binary operations on $\{1, 2\}$ is ......

(a) 16 (b) 8 (c) 2 (d) 4

(11) The number of binary operations on $\{1, 2, 3, ..., n\}$ is ......

(a) 2^n (b) n^{n^2} (c) n^3 (d) n^{2n}

(12) If $a * b = a + b + ab$ on $R - \{-1\}$, then a^{-1} is ......

(a) a^3 (b) $\frac{1}{a}$ (c) $\frac{-a}{a+1}$ (d) $\frac{1}{a^2}$

(13) For $a * b = a + b + 10$ on Z, the identity is ......

(a) 0 (b) -5 (c) -10 (d) 1

(14) The number of commutative binary operations on $\{1, 2\}$ is ......

(a) 8 (b) 4 (c) 16 (d) 27

(15) If $a * b = \frac{ab}{100}$ on Q^+, inverse of 0.1 is ......

(a) 100000 (b) 10000 (c) 1000 (d) 10

Section B (2 marks)

(16) $A = [-1, 1]$, $B = [0, 1]$, $C = [-1, 0]$

$S_1 = \{(x, y) \mid x^2 + y^2 = 1, x \in A, y \in A\}$

$S_2 = \{(x, y) \mid x^2 + y^2 = 1, x \in A, y \in B\}$

$S_3 = \{(x, y) \mid x^2 + y^2 = 1, x \in A, y \in C\}$

$S_4 = \{(x, y) \mid x^2 + y^2 = 1, x \in B, y \in C\}$, then

(a) S_1 is not a graph of a function. (b) S_2 is not a graph of a function.

(c) S_3 is not a graph of a function. (d) S_4 is not a graph of a function.

(17) $f : R \to R$, $f(x) = 3^x + 3^{|x|} =$ ......

(a) one-one and onto (b) one-one but not onto

(c) many-one and onto (d) many-one and not onto

(18) $f : R - \{q\} \to R - \{1\}$, $f(x) = \frac{x-p}{x-q}$, $p \neq q$, then f is ......

(a) one-one and onto (b) many-one and not onto

(c) one-one and not onto (d) many-one and onto

(19) $f : [-1, 1] \to [-1, 1], f(x) = -x\,|x|$ is ......

(a) one-one and onto (b) many-one and onto

(c) many-one and not onto (d) one-one and not onto

(20) If $f : R \to R, f(x) = 2x - 3$, then...

(a) $f^{-1}(x) = \frac{1}{2x-3}$ (b) $f^{-1}(x) = \frac{x+3}{2}$

(c) f^{-1} does not exist (d) $f^{-1}(x) = 3x - 2$

(21) $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is a bijection if...

(a) $f(x) = |x|$ (b) $f(x) = sinx$ (c) $f(x) = x^2$ (d) $f(x) = cosx$

(22) $f : R \to R, f(x) = x^2 + 2x + 3$ is...

(a) a bijection (b) one-one but not onto

(c) onto but not one-one (d) many-one and not onto

(23) If $a * b = ab + 1$ on R, is...

(a) commutative, but not associative (b) associative, but not commutative

(c) neither commutative nor associative (d) both commutative and associative

(24) If $a * b = a^2 + b^2$ on Z, then $*$ is...

(a) commutative and associative (b) commutative and not associative

(c) not commutative and associative (d) neither commutative nor associative

(25) If $a * b = a + b - ab$ on $Q - \{1\}$, then the identity and the inverse of a for $*$ are respectively...

(a) 0 and $\frac{a}{a-1}$ (b) 1 and $\frac{a-1}{a}$ (c) -1 and a (d) 0, $\frac{1}{a}$

(26) If $a * b = \frac{ab}{3}$ on Q^+, then $3 * \left(\frac{1}{5} * \frac{1}{2}\right)$ is...

(a) $\frac{5}{160}$ (b) $\frac{1}{30}$ (c) $\frac{3}{160}$ (d) $\frac{3}{60}$

(27) If Δ is defined on P(X) $(X \neq \emptyset)$ by, $A \Delta B = (A \cup B) - (A \cap B)$, then...

(a) identity for Δ is $\emptyset$ and inverse of A is A

(b) identity for Δ is A and inverse of A is $\emptyset$

(c) identity for Δ is A' and inverse of A is A

(d) identity for Δ is X and inverse of A is $\emptyset$

Section C (3 marks)

(28) S is defined on $N \times N$ by $((a, b), (c, d)) \in S \Leftrightarrow a + d = b + c$...

(a) S is reflexive, but not symmetric (b) S is reflexive and transitive only

(c) S is an equivalence relation (d) S is transitive only

(29) Let S be the relation on the set A = {5, 6, 7, 8},

S = {(5, 6), (6, 6), (5, 5), (8, 8), (5, 7), (7, 7), (7, 6)}, then...

(a) S is reflexive and symmetric but not transtive

(b) S is reflexive and transitive but not symmetric

(c) S is symmetric and transitive but not reflexive

(d) S is an equivalence relation.

(30) If $f : R^+ \to R$, $f(x) = \frac{x}{x+1}$ is ......

(a) one-one and onto (b) one-one and not onto

(c) not one-one and not onto (d) onto but not one-one

(31) If $f : R \to R$, $f(x) = [x]$, $g : R \to R$, $g(x) = sinx$, $h : R \to R$, $g(x) = 2x$, then $ho(gof)$ = ...

(a) $sin[x]$ (b) $[sin2x]$ (c) $2(sin[x])$ (d) $sin2[x]$

(32) If $f : R \to (-1, 1)$, $f(x) = \frac{-x|x|}{1+x^2}$, then f^{-1} = ...

(a) $\frac{1}{x^2+1}$ (b) $-$signum $x\sqrt{\frac{|x|}{1-|x|}}$

(c) $-\frac{\sqrt{x}}{1-x}$ (d) $\frac{x^2}{x^2+1}$

(33) $f : R \to R$, $f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$

$g : R \to R$, $g(x) = 1 + x - [x]$, then for all x, $f(g(x))$ = ...

(a) 1 (b) 2 (c) 0 (d) -1

Section D (4 marks)

(34) If $f : \{x \mid x \geq 1, x \in R\} \to \{x \mid x \geq 2, x \in R\}$, $f(x) = x + \frac{1}{x}$, $f^{-1}(x)$ = ...

(a) $\frac{x+\sqrt{x^2-4}}{2}$ (b) $\frac{x-\sqrt{x^2-4}}{2}$ (c) $\frac{x^2+1}{x}$ (d) $\sqrt{x^2-4}$

(35) If $f : R \to R$, $f(x) = x - [x]$, then $f^{-1}(x)$ = ...

(a) does not exist (b) is x (c) is $[x]$ (d) $x - [x]$

(36) If $f(x) = \frac{x}{\sqrt{1+x^2}}$, then $(fo(fof))(x)$ = ...

(a) $\frac{x}{1+x^2}$ (b) $\frac{1+x^2}{x}$ (c) $\frac{x}{\sqrt{1+2x^2}}$ (d) $\frac{x}{\sqrt{1+3x^2}}$

(37) $f : R \to R$, $f(x) = x^2$, $g : R \to R$, $g(x) = 2^x$, then $\{x \mid (fog)(x) = (gof)(x)\}$ = ...

(a) {0} (b) {0, 1} (c) R (d) {0, 2}

(38) $f : R \rightarrow Z, f(x) = [x]$ is ......

(a) one-one and onto and has an inverse (b) many-one and not onto, no inverse

(c) many-one and onto, no inverse (d) one-one and not onto, no inverse

(39) A = {0, 1, 2, 3, 4, 5, 6}. If $a, b \in$ A, $a * b$ = remainder when ab is divided by 7. From binary operation table of *, inverse of 2 is ......

(a) 1 (b) 5 (c) 6 (d) 4

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Summary

We have studied the following points in this chapter :

1. Relation and equivalence relation.
2. One-one and onto functions
3. Composition of functions
4. Inverse of a function
5. Binary Operations on a set

Srinivasa Ramanujan

Born in Erode, Madras Presidency, to a poor Brahmin family, Ramanujan first encountered formal mathematics at age 10. He demonstrated a natural ability, and was given books on advanced trigonometry written by S. L. Loney. He mastered them by age 12, and even discovered theorems of his own, including independently re-discovering Euler's identity. He demonstrated unusual mathematical skills at school, winning accolades and awards. By 17, Ramanujan conducted his own mathematical research on Bernoulli numbers and the Euler–Mascheroni constant. He received a scholarship to study at Government College in Kumbakonam, but lost it when he failed his non-mathematical coursework. He joined another college to pursue independent mathematical research, working as a clerk in the Accountant-General's office at the Madras Port Trust Office to support himself. In 1912–1913, he sent samples of his theorems to three academics at the University of Cambridge. Only Hardy recognised the brilliance of his work, subsequently inviting Ramanujan to visit and work with him at Cambridge. He became a Fellow of the Royal Society and a Fellow of Trinity College, Cambridge, dying of illness, malnutrition and possibly liver infection in 1920 at the age of 32.

During his short lifetime, Ramanujan independently compiled nearly 3900 results (mostly identities and equations). Although a small number of these results were actually false and some were already known, most of his claims have now been proven correct. He stated results that were both original and highly unconventional, such as the Ramanujan prime and the Ramanujan theta function, and these have inspired a vast amount of further research. However, the mathematical mainstream has been rather slow in absorbing some of his major discoveries. The Ramanujan Journal, an international publication, was launched to publish work in all areas of mathematics influenced by his work.