INVERSE TRIGONOMETRIC FUNCTIONS 2

No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it untill it also gives the impression of being beautiful.

– George Boole

Mathematics consists of proving the most obvious things in the least obvious way.

– George Polya

2.1 Introduction

We have studied that a function has an inverse if and only if it is one-one and onto. There are many functions which are not one-one or not onto or both and hence they cannot have an inverse function. In class XI, we have studied that all trigonometric functions are periodic and hence they are all many-one functions. Therefore, they cannot have an inverse. In order to have inverse of these functions, we must restrict their domain and codomain in such a way that they become one-one and onto. With this modified domain and codomain, it can have an inverse.

We know that if $f = \{(x, y) \mid y = f(x), x \in A, y \in B\}$ is one-one and onto, then f^{-1} exists and $f^{-1} = \{(y, x) \mid y = f(x), x \in A, y \in B\}$

Also $fof^{-1} = I_B$ and $f^{-1}of = I_A$

$\therefore \quad x \in A \Rightarrow (f^{-1}of)(x) = x, \ y \in B \Rightarrow (fof^{-1})(y) = y$

In this chapter, we shall discuss the existence of the inverse of trigonometric functions and discuss their properties.

2.2 Inverse of *sine* Function

We know that $sin : R \to R$ is many-one and range of *sine* is $[-1, 1]$. So, it is not onto R. $sine = \{(x, y) \mid y = sinx, x \in R, y \in [-1, 1]\}$ is a many-one function on R and is onto $[-1, 1]$. It is many-one and periodic with period 2π. We can see from its graph that, if the domain of *sine* is taken as, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ or $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ or $\left[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right]$, $k \in Z$, it becomes one-one and remains onto $[-1, 1]$.

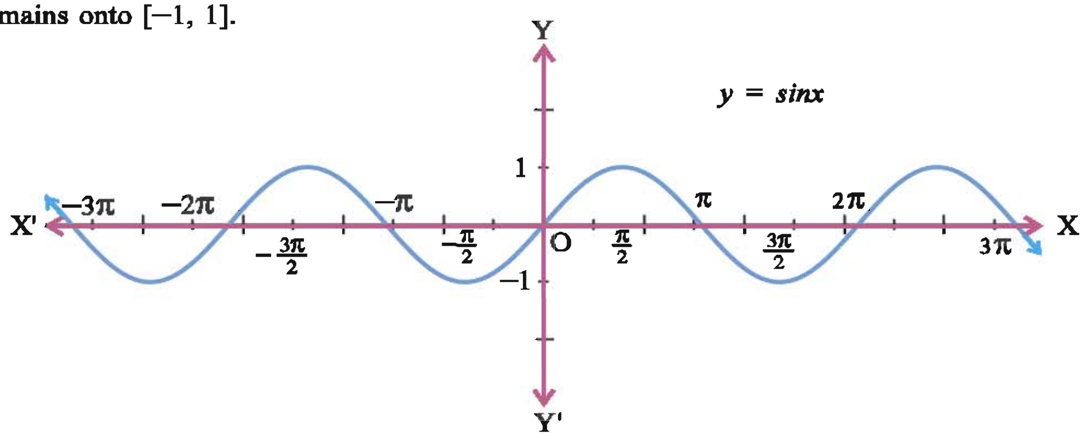

Figure 2.1

So, to define the inverse of *sine* function, we can take any of these intervals as the domain of *sine*. We shall take the domain of *sine* function as $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to define the inverse of *sine*. So we consider the function $sin = \left\{(x, y) \mid y = sinx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \in [-1, 1]\right\}$. **This is a one-one and onto function. Therefore, it will have an inverse function. The inverse of *sine* function is denoted by sin^{-1}.**

$\therefore \quad sin^{-1} = \left\{(y, x) \mid y = sinx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \in [-1, 1]\right\}$.

Thus, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1, 1]$

$\therefore \quad y = sinx \Leftrightarrow sin^{-1}y = x$

The domain of sin^{-1} is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Remember that if $y \in [-1, 1]$, $sin^{-1}y$ is not just any real x for which $sinx = y$ but only that $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $sinx = y$. For instance, given $y = \frac{\sqrt{3}}{2}$, we know that $sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. Although $sin\left(\pi - \frac{\pi}{3}\right) = sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, we can not write $sin^{-1}\frac{\sqrt{3}}{2} = \frac{2\pi}{3}$ because $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Also $\forall\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $sin^{-1}(sin\theta) = \theta$.

$sin(sin^{-1}x) = x$. $\forall x \in [-1, 1]$.

For instance, $sin\left(sin^{-1}\frac{5}{7}\right) = \frac{5}{7}$, because $\frac{5}{7} \in [-1, 1]$. $sin^{-1}\left(sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$ because $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $sin^{-1}\left(sin\left(\frac{3\pi}{5}\right)\right) \neq \frac{3\pi}{5}$ because $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

If the inverse of $f : A \to B$ is $f^{-1} : B \to A$, then we know that,

$$fof^{-1} = I_B \text{ and } f^{-1}of = I_A$$

Thus, $sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ has inverse, $sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore \quad sin^{-1}(sinx) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $sin(sin^{-1}x) = x \quad \forall x \in [-1, 1]$.

We note that,

(1) $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftrightarrow -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \Leftrightarrow |x| \leq \frac{\pi}{2}$ and

$y \in [-1, 1] \Leftrightarrow -1 \leq y \leq 1 \Leftrightarrow |y| \leq 1$.

(2) $sin^{-1}x \neq \frac{1}{sinx}$, that is $sin^{-1}x \neq (sinx)^{-1}$

2.3 The Graph of $y = sin^{-1}x$

The domain and the range of sin^{-1} are $[-1, 1]$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ respectively. Its graph will be confined between the two vertical lines $x = -1$ and $x = 1$ and the two horizontal lines $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

We can use our knowledge of the graph of $y = sinx$ to get the graph of $y = sin^{-1}x$. To obtain it, let us first examine how to find the graph of f^{-1} from the graph of f, when inverse of f exists. The graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$ are very interestingly related. If point (a, b) is on the graph of $y = f(x)$, then $b = f(a)$ and so $a = f^{-1}(b)$. Therefore, the point (b, a) is on the graph of $y = f^{-1}(x)$. The converse is also true. Hence, $A(a, b)$ is on the graph of $y = f(x)$ if and only if $B(b, a)$ is on the graph of $y = f^{-1}(x)$.

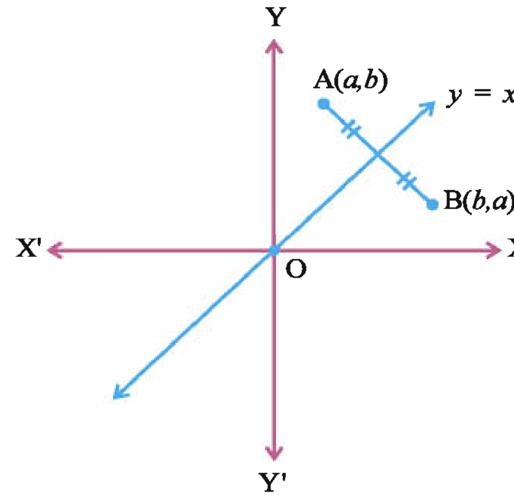

Figure 2.2

We can see that the line $y = x$ is the perpendicular bisector of the line-segment joining $A(a, b)$ and $B(b, a)$. Slope of the segment joining $A(a, b)$ and $B(b, a)$ is $\frac{b-a}{a-b} = -1$. The slope of $y = x$ is 1. Hence $\overleftrightarrow{AB}$ is perpendicular to the line $y = x$. Also the mid-point $\overline{AB}$ is $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ and obviously it lies on the line $y = x$.

- The line $y = x$ is perpendicular bisector of $\overline{AB}$. Thus, $B(b, a)$ is the mirror image of $A(a, b)$ in the line $y = x$. Thus, the graph of $y = f^{-1}(x)$ is just the image of the graph of $y = f(x)$ in the line $y = x$.

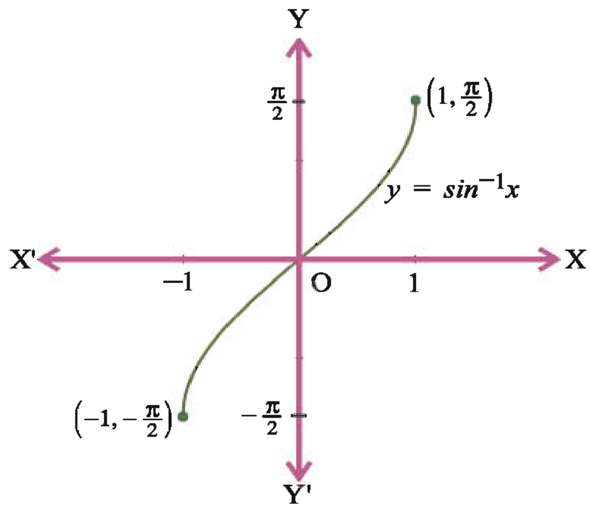

Figure 2.3

Thus, the graph of $y = sin^{-1}x$ is obtained by simply reflecting the graph of *sin* through the line $y = x$. First draw the graph of $y = sinx$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \in [-1, 1]$ on a piece of paper. Now fold this paper on the line $y = x$. Now turn the paper upside down, interchange the X-axis and Y-axis and look at the graph. What you see is the graph of $y = sin^{-1}x$.

Note : The student herself should perform this activity in the class-room.

For the graph of $y = sinx$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \in [-1, 1]$ and for the graph of $y = sin^{-1}x$, $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Example 1 : Obtain the value of : (1) $sin^{-1}\left(\frac{1}{2}\right)$, (2) $sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, (3) $sin^{-1}\left(-\frac{1}{2}\right)$.

Solution : (1) $sin^{-1}\left(\frac{1}{2}\right) = sin^{-1}\left(sin\frac{\pi}{6}\right) = \frac{\pi}{6}$, because $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(2) $sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = sin^{-1}\left(sin\frac{\pi}{4}\right) = \frac{\pi}{4}$, because $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(3) $sin^{-1}\left(-\frac{1}{2}\right)$. $= sin^{-1}\left(-sin\frac{\pi}{6}\right) = sin^{-1}\left(sin\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$, because $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2.4 Inverse of *cosine* Function

We know that $cos : R \to R$ is many-one and range of *cosine* is $[-1, 1]$. So, it is not onto. $cos = \{(x, y) \mid y = cosx, x \in R, y \in [-1, 1]\}$ is a many-one function onto $[-1, 1]$ with

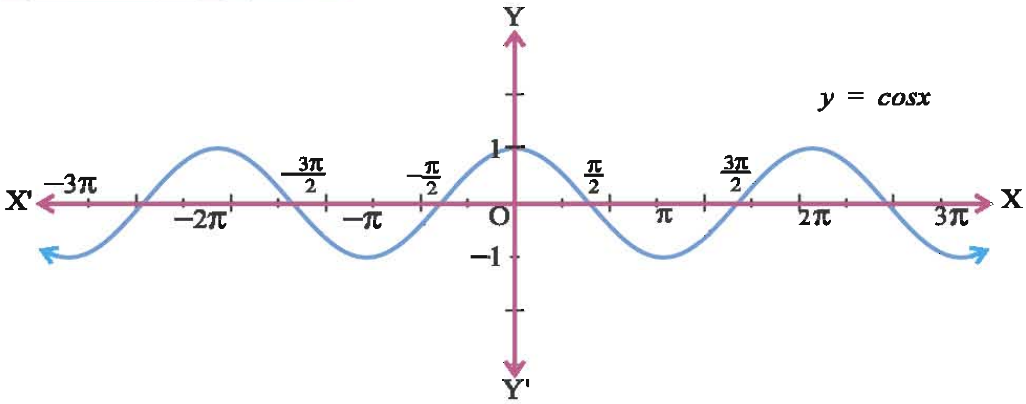

Figure 2.4

period 2π. We see from its graph that it becomes one-one and onto if the domain is restricted to $[0, \pi]$ or $[\pi, 2\pi]$ or $[2\pi, 3\pi]$ or... $[k\pi, (k + 1)\pi]$, $k \in Z$

We shall take the domain of *cosine* function as $[0, \pi]$ to define the inverse of *cosine*. So consider the function **$cos = \{(x, y) \mid y = cosx, x \in [0, \pi], y \in [-1, 1]\}$. This is a one-one and onto function. So, its inverse exists. We denote its inverse by cos^{-1}.**

$\therefore$ $cos^{-1} = \{(y, x) \mid y = cosx, x \in [0, \pi], y \in [-1, 1]\}$. Thus, for $x \in [0, \pi]$ and $y \in [-1, 1]$, $y = cosx \Leftrightarrow cos^{-1}y = x$.

The domain of cos^{-1} is $[-1, 1]$ and its range is $[0, \pi]$.

Like *sine* function, here also we have to remember that if $y \in [-1, 1]$. $cos^{-1}y$ is not just any real x for which $cosx = y$ but only that $x \in [0, \pi]$, for which $cosx = y$. For instance $cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{6} \in [0, \pi]$. Hence, $cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. But, $cos\left(-\frac{\pi}{6}\right) = cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. But, $-\frac{\pi}{6} \notin [0, \pi]$.

$\therefore$ $cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \neq -\frac{\pi}{6}$.

$cos : [0, \pi] \to [-1, 1]$ has the inverse $cos^{-1} : [-1, 1] \to [0, \pi]$.

So, $cos^{-1}(cosx) = x, \forall x \in [0, \pi]$ and $cos(cos^{-1}x) = x, \forall x \in [-1, 1]$.

Note that $sin^{-1}(sinx)$ and $cos^{-1}(cosx)$ exist, $\forall x \in R$, but they may not be equal to x. However, each will be equal to x in its appropriate domains. [The above experiment can be done with some appropriate change.]

2.5 The Graph of $y = cos^{-1}x$

We have discussed the method of drawing the graph of the inverse function from the graph of the function. As in the case of the graph of sin^{-1} the graph of cos^{-1} as obtained from the graph of $y = cosx$, $x \in [0, \pi]$ is shown in the figure 2.5.

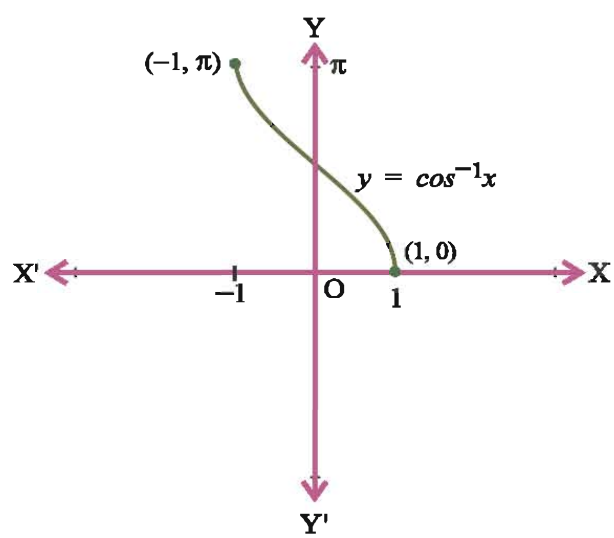

Figure 2.5

Example 2 : Obtain the value of : (1) $cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (2) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution : (1) $cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = cos^{-1}\left(cos\frac{\pi}{4}\right) = \frac{\pi}{4}$, because $\frac{\pi}{4} \in [0, \pi]$.

(2) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = cos^{-1}\left(cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$, because $\frac{5\pi}{6} \in [0, \pi]$.

2.6 Inverse of *tan* Function

We know that $tan : R - \left\{(2k + 1)\frac{\pi}{2} \mid k \in Z\right\} \to R$ is many-one and range of *tan* is R. So it is onto.

$tan = \left\{(x, y) \mid y = tanx, x \in R - \left\{(2k + 1)\frac{\pi}{2} \mid k \in Z\right\}, y \in R\right\}$ is many-one function with period π and it is onto R. If its domain is restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ or $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ or $\left((2k - 1)\frac{\pi}{2}, (2k + 1)\frac{\pi}{2}\right)$, $k \in Z$; it becomes one-one and remains onto R. So we can get its inverse by taking one of these intervals as its domain. We shall take $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as the domain and get the inverse which is denoted by tan^{-1}. So, $tan^{-1} = \left\{(y, x) \mid y = tanx, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), y \in R\right\}$.

Thus, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $y \in R$,

$$y = tanx \Leftrightarrow tan^{-1}y = x.$$

Domain of tan^{-1} is R and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Note : $tan^{-1}x \neq (tanx)^{-1}$ i.e. $tan^{-1}x \neq \frac{1}{tanx}$. $tan^{-1}x \neq \frac{sin^{-1}x}{cos^{-1}x}$.

$tan^{-1}(tanx) = x, \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan(tan^{-1}x) = x, \forall x \in R$.

$tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

So, $tan^{-1}(-1) = -\frac{\pi}{4}$

But $tan\left(\frac{3\pi}{4}\right) = -1$ does not imply $tan^{-1}(-1) = \frac{3\pi}{4}$ as $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$tan^{-1}\left(tan\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$, because $-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan\left(tan^{-1}\left(\frac{533}{413}\right)\right) = \frac{533}{413}$.

But $tan^{-1}\left(tan\frac{5\pi}{6}\right) \neq \frac{5\pi}{6}$ because $\frac{5\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

2.7 The Graph of $y = tan^{-1}x$

The graph of $y = tan^{-1}x$ is obtained by taking the image of $y = tanx$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $y \in R$ in the line $y = x$. We get the graph of $y = tan^{-1}x$ as shown.

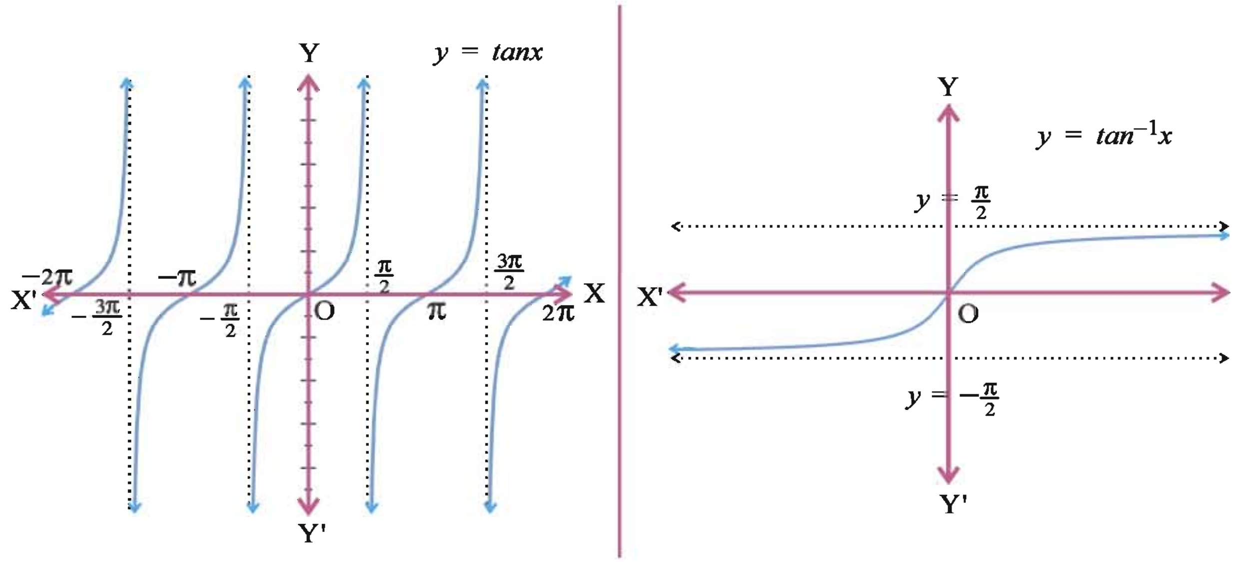

Figure 2.6

2.8 Inverse of *cot* Function

We know that $cot : R - \{k\pi \mid k \in Z\} \rightarrow R$ is many-one and the range of *cot* is R. So *cot* is onto. $cot = \{(x, y) \mid y = cotx, x \in R - \{k\pi \mid k \in Z\}, y \in R\}$ is a many-one, onto and periodic function with period π. The function becomes one-one and onto R, if its domain is restricted to $(0, \pi)$ or $(\pi, 2\pi)$ or $(2\pi, 3\pi)$ or $(k\pi, (k + 1)\pi)$, $k \in Z$. We shall take the domain as $(0, \pi)$ and get the inverse which is denoted by cot^{-1}.

So, $cot^{-1} = \{(y, x) \mid y = cotx, x \in (0, \pi), y \in R\}$.

Thus, for $x \in (0, \pi)$ and $y \in R$,

$$y = cotx \Leftrightarrow cot^{-1}y = x.$$

Domain of cot^{-1} is R and its range is $(0, \pi)$.

$cot^{-1}(cotx) = x, x \in (0, \pi)$ and $cot(cot^{-1}x) = x, x \in R$.

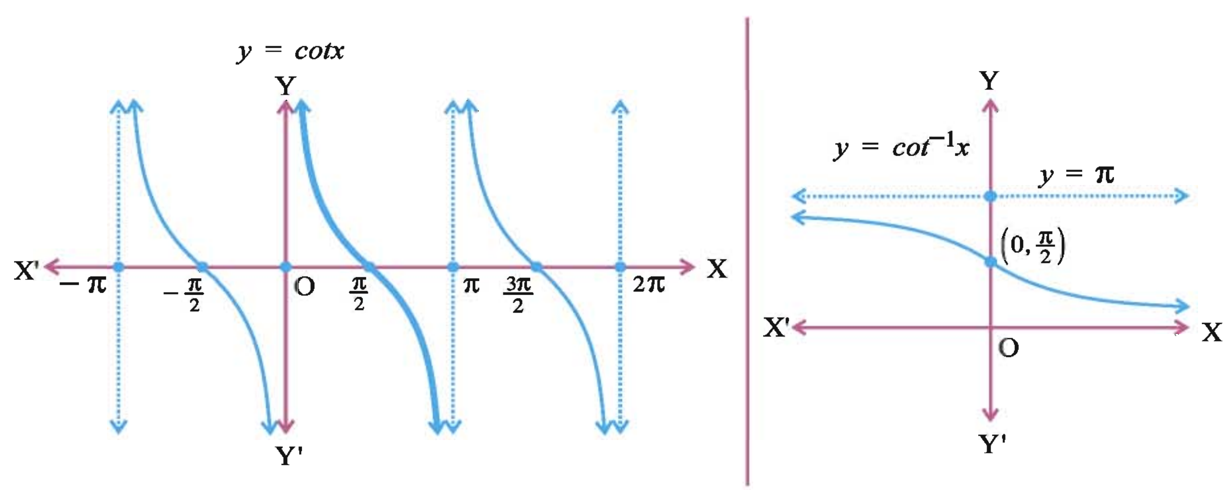

Figure 2.7

Note that $cot^{-1}\left(cot\left(\frac{3\pi}{4}\right)\right) = \frac{3\pi}{4}$, because $\frac{3\pi}{4} \in (0, \pi)$

Also $cot\left(\frac{3\pi}{4}\right) = -1 \Leftrightarrow cot^{-1}(-1) = \frac{3\pi}{4}$.

$cot\left(-\frac{\pi}{4}\right) = -1$, but $cot^{-1}(-1) \neq -\frac{\pi}{4}$ because $-\frac{\pi}{4} \notin (0, \pi)$.

$cot^{-1}\left(cot\frac{4\pi}{3}\right) \neq \frac{4\pi}{3}$, because $\frac{4\pi}{3} \notin (0, \pi)$.

However, $cot\left(\frac{4\pi}{3}\right) = cot\left(\pi + \frac{\pi}{3}\right) = cot\frac{\pi}{3}$ and $\frac{\pi}{3} \in (0, \pi)$.

So, $cot^{-1}\left(cot\frac{4\pi}{3}\right) = cot^{-1}\left(cot\frac{\pi}{3}\right) = \frac{\pi}{3}$.

The graphs of $y = cotx$ and $y = cot^{-1}x$ are given in figure 2.7.

2.9 The Inverse of *sec* Function

We know that $cos : [0, \pi] \to [-1, 1]$ is one-one and onto.

$\therefore \quad sec : [0, \pi] - \left\{\frac{\pi}{2}\right\} \to R - (-1, 1)$ is also one-one and onto.

$\therefore \quad sec = \left\{(x, y) \mid y = secx, x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}, y \in R - (-1, 1)\right\}$ is one-one and onto.

Therefore inverse of this function exists and is denoted by sec^{-1}.

So, $sec^{-1} = \left\{(y, x) \mid y = secx, x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}, y \in R - (-1, 1)\right\}$.

Thus, for $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}, y \in R - (-1, 1), y = secx \Leftrightarrow sec^{-1}y = x$.

Domain of sec^{-1} is $R - (-1, 1)$ and its range is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

Also, $sec\left(\frac{\pi}{4}\right) = \sqrt{2}$.

So, $sec^{-1}(\sqrt{2}) = \frac{\pi}{4}$, because $\frac{\pi}{4} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

But $sec\left(-\frac{\pi}{4}\right) = \sqrt{2}$ does not imply $sec^{-1}(\sqrt{2}) = -\frac{\pi}{4}$ because $-\frac{\pi}{4} \notin [0, \pi] - \left\{\frac{\pi}{2}\right\}$.

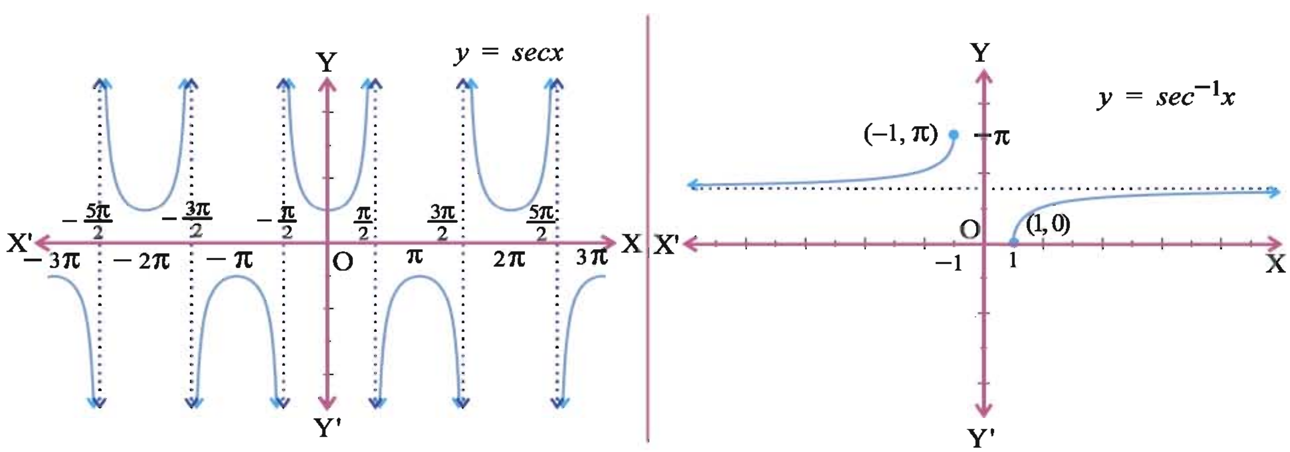

Figure 2.8

For each x, $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, $sec^{-1}(secx) = x$ and for each $x \in R - (-1, 1)$, $sec(sec^{-1}x) = x$.

We note that $x \in R - (-1, 1) \Leftrightarrow x \leq -1$ or $x \geq 1 \Leftrightarrow |x| \geq 1$.

The graph of $y = secx$ and $y = sec^{-1}x$ are given in figure 2.8.

2.10 Inverse of *cosec* Function

We know that $sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is one-one and onto.

$\therefore$ $cosec : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \to R - (-1, 1)$ is also one-one and onto.

$\therefore$ $cosec = \left\{(x, y) \mid y = cosecx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}, y \in R - (-1, 1)\right\}$ is one-one and onto.

Therefore, the inverse of this function exists and is denoted by $cosec^{-1}$.

So, $cosec^{-1} = \left\{(y, x) \mid y = cosecx, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}, y \in R - (-1, 1)\right\}$.

Thus, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}, y \in R - (-1, 1)$.

$$y = cosecx \Leftrightarrow cosec^{-1}y = x.$$

Domain of $cosec^{-1}$ is $R - (-1, 1)$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Also, $cosec\frac{\pi}{3} = \frac{2}{\sqrt{3}}$, $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

So, $cosec^{-1}\frac{2}{\sqrt{3}} = \frac{\pi}{3}$.

For each $x \in R - (-1, 1)$, $cosec(cosec^{-1}x) = x$ and for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $cosec^{-1}(cosecx) = x$.

The graphs of $y = cosecx$ and $y = cosec^{-1}x$ are given in figure 2.9.

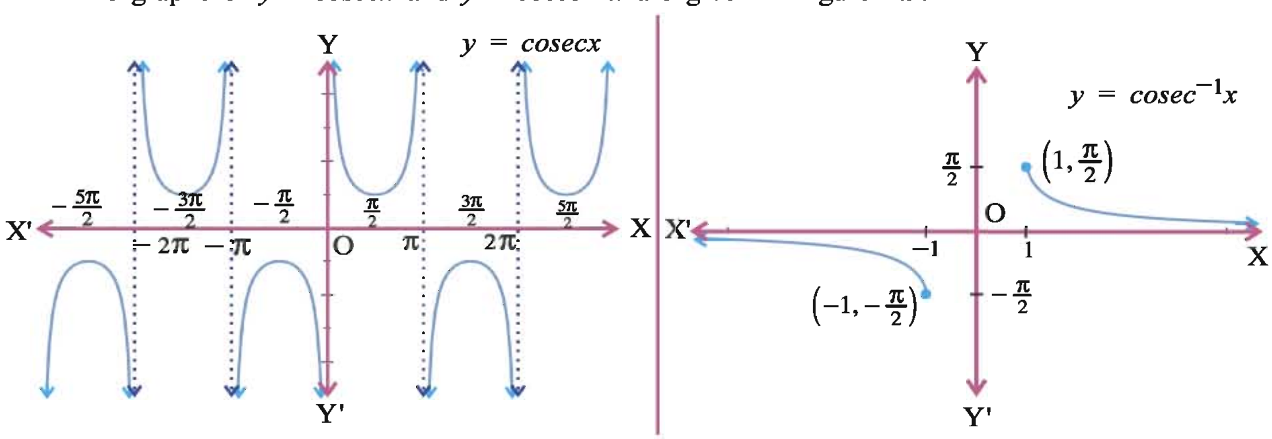

Figure 2.9

Example 3 : Evaluate : **(1)** $tan^{-1}(\sqrt{3})$ **(2)** $cot^{-1}(-\sqrt{3})$ **(3)** $cosec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Solution : (1) $tan^{-1}(\sqrt{3}) = tan^{-1}\left(tan\frac{\pi}{3}\right) = \frac{\pi}{3}$ $\left(\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(2) $cot^{-1}(-\sqrt{3}) = cot^{-1}\left(-cot\frac{\pi}{6}\right) = cot^{-1}\left(cot\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ $\left(\frac{5\pi}{6} \in (0, \pi)\right)$

(3) $cosec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = cosec^{-1}\left(-cosec\frac{\pi}{3}\right) = cosec^{-1}\left(cosec\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$ $\left(-\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}\right)$

Example 4 : Evaluate : (1) $cos^{-1}\left(cos\frac{2\pi}{3}\right)$ (2) $sin^{-1}\left(sin\frac{2\pi}{3}\right)$ (3) $tan^{-1}\left(tan\frac{3\pi}{4}\right)$

(4) $cot^{-1}\left(tan\frac{7\pi}{4}\right)$ (5) $cos^{-1}\left(sin\frac{\pi}{5}\right)$

Solution : (1) $cos^{-1}\left(cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ $\left(\frac{2\pi}{3} \in [0, \pi]\right)$

(2) $sin^{-1}\left(sin\frac{2\pi}{3}\right) = sin^{-1}\left(sin\left(\pi - \frac{\pi}{3}\right)\right)$ $\left(\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$= sin^{-1}\left(sin\frac{\pi}{3}\right)$

$\therefore\ sin^{-1}\left(sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$ $\left(\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

(3) $tan^{-1}\left(tan\frac{3\pi}{4}\right) = tan^{-1}\left(tan\left(\pi - \frac{\pi}{4}\right)\right)$

$= tan^{-1}\left(-tan\left(\frac{\pi}{4}\right)\right)$

$= tan^{-1}\left(tan\left(-\frac{\pi}{4}\right)\right)$

$\therefore\ tan^{-1}\left(tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$ $\left(-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

(4) $cot^{-1}\left(tan\frac{7\pi}{4}\right) = cot^{-1}\left(tan\left(2\pi - \frac{\pi}{4}\right)\right)$

$= cot^{-1}\left(-tan\frac{\pi}{4}\right)$

$= cot^{-1}\left(cot\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right)$

$= cot^{-1}\left(cot\frac{3\pi}{4}\right)$

$\therefore\ cot^{-1}\left(tan\frac{7\pi}{4}\right) = \frac{3\pi}{4}$ $\left(\frac{3\pi}{4} \in (0, \pi)\right)$

(5) $cos^{-1}\left(sin\frac{\pi}{5}\right) = cos^{-1}\left(cos\left(\frac{\pi}{2} - \frac{\pi}{5}\right)\right)$

$= cos^{-1}\left(cos\frac{3\pi}{10}\right)$

$\therefore\ cos^{-1}\left(sin\frac{\pi}{5}\right) = \frac{3\pi}{10}$ $\left(\frac{3\pi}{10} \in [0, \pi]\right)$

Example 5 : Find the value of :

(1) $cos\left(2sin^{-1}\frac{3}{4}\right)$ (2) $sin\left(2tan^{-1}\frac{4}{5}\right)$ (3) $tan^2\left(\frac{1}{2}cos^{-1}\frac{3}{4}\right)$ (4) $cos\left(3cos^{-1}\frac{2}{3}\right)$

Solution : (1) Consider $cos\left(2sin^{-1}\frac{3}{4}\right)$.

Let $sin^{-1}\frac{3}{4} = \theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So $sin\theta = \frac{3}{4}$.

So, $cos\left(2sin^{-1}\frac{3}{4}\right) = cos2\theta$

$= 1 - 2sin^2\theta = 1 - 2\left(\frac{9}{16}\right) = -\frac{1}{8}$

$\therefore \quad cos\left(2sin^{-1}\frac{3}{4}\right) = -\frac{1}{8}$

(2) Let $tan^{-1}\frac{4}{5} = \theta$. $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $tan\theta = \frac{4}{5}$

So, $sin\left(2tan^{-1}\frac{4}{5}\right) = sin2\theta = \frac{2tan\theta}{1+tan^2\theta} = \frac{2\left(\frac{4}{5}\right)}{1+\frac{16}{25}} = \frac{40}{41}$

$\therefore \quad sin\left(2tan^{-1}\frac{4}{5}\right) = \frac{40}{41}$

(3) Let $cos^{-1}\frac{3}{4} = \theta$, $\theta \in [0, \pi]$. Then $\frac{3}{4} = cos\theta$

So, $tan^2\left(\frac{1}{2}cos^{-1}\frac{3}{4}\right) = tan^2\left(\frac{\theta}{2}\right) = \frac{1-cos\,\theta}{1+cos\,\theta} = \frac{1-\frac{3}{4}}{1+\frac{3}{4}} = \frac{4-3}{4+3} = \frac{1}{7}$

$\therefore \quad tan^2\left(\frac{1}{2}cos^{-1}\frac{3}{4}\right) = \frac{1}{7}$

(4) Let $cos^{-1}\frac{2}{3} = \theta$, $\theta \in [0, \pi]$. Then $cos\theta = \frac{2}{3}$

So, $cos\left(3cos^{-1}\frac{2}{3}\right) = cos3\theta$

$= 4cos^3\theta - 3cos\theta = 4\left(\frac{8}{27}\right) - 3\left(\frac{2}{3}\right) = \frac{32-54}{27} = -\frac{22}{27}$

$\therefore \quad cos\left(3cos^{-1}\frac{2}{3}\right) = -\frac{22}{27}$

Example 6 : Express the following in the simplest form :

(1) $tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right)$, $-\pi < x < \pi$ **(2)** $tan^{-1}\left(\frac{cosx}{1+sinx}\right)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution : (1) $tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = tan^{-1}\left(\sqrt{tan^2\frac{x}{2}}\right) = tan^{-1}\left(\left|tan\,\frac{x}{2}\right|\right)$

Case 1 : If $-\pi < x < 0$, then $-\frac{\pi}{2} < \frac{x}{2} < 0$

$\therefore \quad tan\frac{x}{2} < 0$

$\therefore \quad tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = tan^{-1}\left(-tan\frac{x}{2}\right) = tan^{-1}\left(tan\left(-\frac{x}{2}\right)\right)$

Now, $0 < -\frac{x}{2} < \frac{\pi}{2}$. So, $-\frac{\pi}{2} < -\frac{x}{2} < \frac{\pi}{2}$

$\therefore \quad tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = -\frac{x}{2}$

Case 2 : If $0 \le x < \pi$, then $0 \le \frac{x}{2} < \frac{\pi}{2}$

$\therefore \quad tan\frac{x}{2} \ge 0$

$$tan^{-1}\left(\left|tan\frac{x}{2}\right|\right) = tan^{-1}\left(tan\frac{x}{2}\right) = \frac{x}{2} \qquad \left(0 \le \frac{x}{2} < \frac{\pi}{2}\right)$$

$$\therefore \quad tan^{-1}\left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = \begin{cases} \frac{x}{2} & 0 \le x < \pi \\ -\frac{x}{2} & -\pi < x < 0 \end{cases}$$

(2) $$tan^{-1}\left(\frac{cosx}{1+sinx}\right) = tan^{-1}\left(\frac{cos^2\frac{x}{2} - sin^2\frac{x}{2}}{cos^2\frac{x}{2} + sin^2\frac{x}{2} + 2sin\frac{x}{2}cos\frac{x}{2}}\right)$$

$$= tan^{-1}\left(\frac{\left(cos\frac{x}{2} + sin\frac{x}{2}\right)\left(cos\frac{x}{2} - sin\frac{x}{2}\right)}{\left(cos\frac{x}{2} + sin\frac{x}{2}\right)^2}\right)$$

$$= tan^{-1}\left(\frac{cos\frac{x}{2} - sin\frac{x}{2}}{cos\frac{x}{2} + sin\frac{x}{2}}\right)$$

$$= tan^{-1}\left(\frac{1 - tan\frac{x}{2}}{1 + tan\frac{x}{2}}\right) \qquad \left(cos\frac{x}{2} \ne 0, \text{ why ?}\right)$$

$$= tan^{-1}\left(tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$$

Now, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Hence $-\frac{\pi}{4} < -\frac{x}{2} < \frac{\pi}{4}$.

$\therefore \quad 0 < \left(\frac{\pi}{4} - \frac{x}{2}\right) < \frac{\pi}{2}$

Thus, $tan^{-1}\left(\frac{cosx}{1+sinx}\right) = tan^{-1}\left(tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) = \frac{\pi}{4} - \frac{x}{2}$

$\therefore \quad tan^{-1}\left(\frac{cosx}{1+sinx}\right) = \frac{\pi}{4} - \frac{x}{2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Exercise 2.1

1. Evaluate :

(1) $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $sin^{-1}\left(-\frac{1}{2}\right)$ (3) $sec^{-1}(-2)$

(4) $tan^{-1}(-\sqrt{3})$ (5) $sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (6) $cosec^{-1}(-\sqrt{2})$

2. Evaluate :

(1) $cos^{-1}\left(sin\frac{\pi}{7}\right)$ (2) $sin^{-1}\left(cos\frac{\pi}{5}\right)$

(3) $tan^{-1}\left(tan\frac{5\pi}{4}\right)$ (4) $sec^{-1}\left(cosec\left(\frac{\pi}{8}\right)\right)$

3. Evaluate :

(1) $sin\left(2tan^{-1}\frac{2}{5}\right)$ (2) $tan^2\left(\frac{1}{2}cos^{-1}\frac{2}{3}\right)$

(3) $sin\left(2cos^{-1}\frac{4}{5}\right)$ (4) $tan^2\left(\frac{1}{2}sin^{-1}\frac{2}{3}\right)$ (5) $sin\left(3\,sin^{-1}\frac{1}{2}\right)$

4. Express in the simplest form :

$$tan^{-1}\left(\frac{cosx - sinx}{cosx + sinx}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$$

*

2.11 Values of Inverse Trigonometric Functions for $-x$

We have seen that by restricting the domain and codomain of a trigonometric function, it can be made one-one and onto, which is the necessary and sufficient condition for a function to have its inverse. Also, we have restricted the domain in such a way that the domain of each trigonometric function contains $\left(0, \frac{\pi}{2}\right)$ as its subset. By doing so, we always have the value of each inverse function in $\left(0, \frac{\pi}{2}\right)$, whenever the value of the function is positive. We also make a note that the domain of all the inverse trigonometric functions are such that x belongs to the domain if and only if $-x$ also belongs to it. This is because the domain is $[-1, 1]$ or R or R $-$ $(-1, 1)$, i.e. $|x| \le 1$ or R or $|x| \ge 1$ respectively. If A is in any of this set, then $x \in A \Leftrightarrow -x \in A$.

The values at x and $-x$ of every trigonometric inverse function are related as shown in the following theorem.

Theorem 2.1 : (1) $sin^{-1}(-x) = -sin^{-1}x, \quad |x| \le 1$

(2) $cos^{-1}(-x) = \pi - cos^{-1}x, \quad |x| \le 1$

(3) $tan^{-1}(-x) = -tan^{-1}x, \quad x \in R$

(4) $cot^{-1}(-x) = \pi - cot^{-1}x, \quad x \in R$

(5) $cosec^{-1}(-x) = -cosec^{-1}x, \quad |x| \ge 1$

(6) $sec^{-1}(-x) = \pi - sec^{-1}x, \quad |x| \ge 1$

Proof : (1) $|x| \le 1$

Suppose $sin^{-1}x = \theta$. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $x = sin\theta$.

$sin(-\theta) = -sin\theta$

$\therefore \quad sin(-\theta) = -x$ (i)

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} \ge -\theta \ge -\frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} \le -\theta \le \frac{\pi}{2}$$

$\therefore \quad -\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $|x| = |-x|$. Hence $|x| \le 1 \Rightarrow |-x| \le 1$

$\therefore$ By (i), $sin(-\theta) = -x$ $\quad \left(-\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], |-x| \le 1\right)$

$\therefore \quad sin^{-1}(-x) = -\theta = -sin^{-1}x$

$\therefore \quad sin^{-1}(-x) = -sin^{-1}x$

(2) Suppose $cos^{-1}x = \theta$. $\theta \in [0, \pi]$, $|x| \le 1$. Then $x = cos\theta$

Also, $cos(\pi - \theta) = -cos\theta$,

$\therefore \quad cos(\pi - \theta) = -x$ (i)

$\theta \in [0, \pi] \Rightarrow 0 \le \theta \le \pi$

$\Rightarrow 0 \ge -\theta \ge -\pi$

$\Rightarrow \pi \ge (\pi - \theta) \ge 0$

$\Rightarrow 0 \le (\pi - \theta) \le \pi$

$\therefore$ $(\pi - \theta) \in [0, \pi]$ and $|x| = |-x|$. Thus $|x| \le 1 \Rightarrow |-x| \le 1$

$\therefore$ By (i), $cos(\pi - \theta) = -x$ $\quad (\pi - \theta \in [0, \pi], |-x| \le 1)$

$\therefore$ $cos^{-1}(-x) = \pi - \theta = \pi - cos^{-1}x$

$\therefore$ $cos^{-1}(-x) = \pi - cos^{-1}x$

(3) Suppose $tan^{-1}x = \theta$. $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $x \in R$. Then $x = tan\theta$

Now, $tan(-\theta) = -tan\theta = -x$ $\quad$ (i)

$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} > -\theta > -\frac{\pi}{2}$

$\therefore$ $-\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $x \in R$. Thus $-x \in R$

$\therefore$ By (i), $tan(-\theta) = -x$ $\quad \left(-\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), -x \in R\right)$

$\therefore$ $tan^{-1}(-x) = -\theta = -tan^{-1}x$

$\therefore$ $tan^{-1}(-x) = -tan^{-1}x$

Similarly we can prove (4), (5) and (6).

Example 7 : Evaluate :

(1) $sin^{-1}\left(-\frac{1}{2}\right)$ (2) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (3) $tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (4) $cot^{-1}(-1)$

Solution : (1) $sin^{-1}\left(-\frac{1}{2}\right) = -sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$

(2) $cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - cos^{-1}\frac{\sqrt{3}}{2} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

(3) $tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(4) $cot^{-1}(-1) = \pi - cot^{-1}1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

2.12 Values of Trigonometric Functions for $\frac{1}{x}$

Now we get relations between the values of trigonometric inverse functions at x and at $\frac{1}{x}$, when $x \ne 0$.

Theorem 2.2 : (1) $cosec^{-1}x = sin^{-1}\frac{1}{x}$, $\quad |x| \ge 1$

(2) $sec^{-1}x = cos^{-1}\frac{1}{x}$, $\quad |x| \ge 1$

(3) (a) $cot^{-1}x = tan^{-1}\frac{1}{x}$, $\quad x > 0$

(b) $cot^{-1}x = tan^{-1}\frac{1}{x} + \pi$, $\quad x < 0$

Proof : (1) Let $cosec^{-1}x = \theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$. Then $x = cosec\theta$. $|x| \ge 1$

$|x| \geq 1$. So $x \neq 0$ and $\left|\frac{1}{x}\right| \leq 1$.

$cosec\theta = x$

$\therefore \quad sin\theta = \frac{1}{x}$

$\therefore \quad \theta = sin^{-1}\frac{1}{x}$ $\quad\quad \left(\theta \in \left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}\right) \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \left|\frac{1}{x}\right| \leq 1\right)$

$\therefore \quad cosec^{-1}x = sin^{-1}\frac{1}{x}$

(2) Let $sec^{-1}x = \theta$, $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, $|x| \geq 1$. Then $x = sec\theta$

$|x| \geq 1$. So $x \neq 0$ and $\left|\frac{1}{x}\right| \leq 1$.

$sec\theta = x$

$\therefore \quad cos\theta = \frac{1}{x}$

$\therefore \quad \theta = cos^{-1}\frac{1}{x}$ $\quad\quad \left(\theta \in \left([0, \pi] - \left\{\frac{\pi}{2}\right\}\right) \subset [0, \pi], \left|\frac{1}{x}\right| \leq 1\right)$

$\therefore \quad sec^{-1}x = cos^{-1}\frac{1}{x}$

(3) **(a)** Let $cot^{-1}x = \theta$, $\theta \in (0, \pi)$, $x \in R$

$\therefore \quad cot\theta = x$

$x > 0$ and hence $x \neq 0$. So $\frac{1}{x} \in R$.

$\therefore \quad tan\theta = \frac{1}{x}$ and $\theta \in (0, \pi)$

Now, since $x > 0$, $tan\theta = \frac{1}{x} > 0$

Also $0 < \theta < \pi$. So we must have $0 < \theta < \frac{\pi}{2}$. $\quad\quad (tan\theta > 0)$

Thus, $tan\theta = \frac{1}{x}$, $\theta \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \theta = tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad cot^{-1}x = tan^{-1}\frac{1}{x}$

(b) As we have seen above, if $cot^{-1}x = \theta$, $\theta \in (0, \pi)$, $x \in R$, then $cot\theta = x$.

Since $x < 0$, $cot\theta = x < 0$. Thus, $tan\theta < 0$ and $\theta \in (0, \pi)$.

This means that $\frac{\pi}{2} < \theta < \pi$

$\therefore \quad \frac{\pi}{2} - \pi < (\theta - \pi) < \pi - \pi$

$\therefore \quad -\frac{\pi}{2} < (\theta - \pi) < 0$

i.e. $\theta - \pi \in \left(-\frac{\pi}{2}, 0\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\frac{1}{x} \in R$ as $x \neq 0$

$tan(\theta - \pi) = tan\theta = \frac{1}{x}$ $\quad\quad$ (Period of *tan* is π)

$\therefore \quad tan(\theta - \pi) = \frac{1}{x}$

$\therefore \quad \theta - \pi = tan^{-1}\frac{1}{x}$ $\quad\quad \left(\theta - \pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \frac{1}{x} \in R\right)$

$\therefore \quad tan^{-1}\frac{1}{x} = cot^{-1}x - \pi$

$\therefore$ For $x < 0$, $cot^{-1}x = tan^{-1}\frac{1}{x} + \pi$.

(**Note :** We can derive from this theorem that

(1) $sin^{-1}x = cosec^{-1}\frac{1}{x}$, $x \in [-1, 1] - \{0\}$

(2) $cos^{-1}x = sec^{-1}\frac{1}{x}$, $x \in [-1, 1] - \{0\}$

(3) **(a)** $tan^{-1}x = cot^{-1}\frac{1}{x}$, $x > 0$

(b) $tan^{-1}x = cot^{-1}\frac{1}{x} - \pi$, $x < 0$

2.13 Formulae for Value of Trigonometric Inverse Functions for Complementary Numbers :

Theorem 2.3 : **(1)** $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$, $|x| \le 1$

(2) $cosec^{-1}x + sec^{-1}x = \frac{\pi}{2}$, $|x| \ge 1$

(3) $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}$, $x \in R$

Proof : (1) Let $sin^{-1}x = \theta$. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $|x| \le 1$. Then $x = sin\theta$

$\therefore$ $cos\left(\frac{\pi}{2} - \theta\right) = x$

Now, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} \ge -\theta \le -\frac{\pi}{2}$

$\Rightarrow \pi \ge \left(\frac{\pi}{2} - \theta\right) \ge 0$

$\Rightarrow 0 \le \left(\frac{\pi}{2} - \theta\right) \le \pi$

$\therefore$ $\left(\frac{\pi}{2} - \theta\right) \in [0, \pi]$ and $|x| \le 1$. Also $cos\left(\frac{\pi}{2} - \theta\right) = x$

$\therefore$ $cos^{-1}x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - sin^{-1}x$

$\therefore$ $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

(2) Let $cosec^{-1}x = \theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $|x| \ge 1$. Then $x = cosec\theta$

$\therefore$ $sec\left(\frac{\pi}{2} - \theta\right) = x$

Now, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$

$\Rightarrow \frac{\pi}{2} \ge -\theta \ge -\frac{\pi}{2}, \theta \ne 0$

$\Rightarrow \pi \ge \left(\frac{\pi}{2} - \theta\right) \ge 0, \theta \ne 0$

$\Rightarrow 0 \le \left(\frac{\pi}{2} - \theta\right) \le \pi, \theta \ne 0$

Also $\frac{\pi}{2} - \theta \ne \frac{\pi}{2}$ as $\theta \ne 0$

$\therefore$ $\frac{\pi}{2} - \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$, $|x| \ge 1$ and $sec\left(\frac{\pi}{2} - \theta\right) = x$.

$\therefore$ $sec^{-1}x = \frac{\pi}{2} - \theta$

$\therefore \quad \theta + sec^{-1}x = \frac{\pi}{2}$

$\therefore \quad cosec^{-1}x + sec^{-1}x = \frac{\pi}{2}$

or we think in another way as, $cosec^{-1}x + sec^{-1}x = sin^{-1}\frac{1}{x} + cos^{-1}\frac{1}{x}$ $\left(|x| \geq 1 \Rightarrow \frac{1}{|x|} \leq 1\right)$

$= \frac{\pi}{2}$ **(By (1))**

(3) can be proved similarly as (1).

2.14 Addition and Subtraction Formulae

Theorem 2.4 : If $x > 0$, $y > 0$, then

(1) $tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

(2) $tan^{-1}x + tan^{-1}y = \pi + tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy > 1$

(3) $tan^{-1}x + tan^{-1}y = \frac{\pi}{2}$, if $xy = 1$

(4) $tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Proof : Here, $x > 0$, $y > 0$.

Let $tan^{-1}x = \alpha$ and $tan^{-1}y = \beta$, $\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad tan\alpha = x > 0$ and $tan\beta = y > 0$

As $tan\alpha$ and $tan\beta$ are positive and $\alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$

(1) $tan(\alpha + \beta) = \frac{tan\alpha + tan\beta}{1 - tan\alpha\, tan\beta} = \frac{x+y}{1-xy}$

Let $x > 0$, $y > 0$ and $xy < 1$. Hence, $(1 - xy) > 0$ and $x + y > 0$.

$\therefore \quad \frac{x+y}{1-xy} > 0$. Hence, $tan(\alpha + \beta) > 0$

Also $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$. $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$

$\therefore \quad 0 < \alpha + \beta < \pi$

But $tan(\alpha + \beta) > 0$. Hence, $\alpha + \beta \in \left(0, \frac{\pi}{2}\right)$

Thus, $tan(\alpha + \beta) = \frac{x+y}{1-xy}$

$\therefore \quad \alpha + \beta = tan^{-1}\frac{x+y}{1-xy}$. $\left((\alpha + \beta) \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\therefore \quad tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$.

(2) $tan(-\pi + \alpha + \beta) = tan(\alpha + \beta)$ **(π is a period of *tan*)**

$= \frac{tan\alpha + tan\beta}{1 - tan\alpha\, tan\beta}$

$\therefore \quad tan(-\pi + \alpha + \beta) = \frac{x+y}{1-xy}$

Now, $x > 0$, $y > 0$. Also, $xy > 1$. So, $1 - xy < 0$

$\therefore \quad \frac{x+y}{1-xy} < 0$

$\therefore \quad tan(-\pi + \alpha + \beta) < 0$

Now $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$.

$\therefore \quad 0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$

$\therefore \quad 0 < \alpha + \beta < \pi$

$\therefore \quad -\pi < \alpha + \beta - \pi < 0$

But, as $tan(-\pi + \alpha + \beta) < 0$, $-\frac{\pi}{2} < \alpha + \beta - \pi < 0$.

So, $\alpha + \beta - \pi \in \left(-\frac{\pi}{2}, 0\right)$

Thus, $tan(-\pi + \alpha + \beta) = \frac{x+y}{1-xy}$, $\alpha + \beta - \pi \in \left(-\frac{\pi}{2}, 0\right)$

$\therefore \quad -\pi + \alpha + \beta = tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\therefore \quad \alpha + \beta = tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi$

$\therefore \quad tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi$

(3) $tan^{-1}x + tan^{-1}y = tan^{-1}x + tan^{-1}\frac{1}{x}$ **($xy = 1$)**

$= tan^{-1}x + cot^{-1}x$ **($x > 0$)**

$= \frac{\pi}{2}$

(4) As we have noted $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$

Thus, $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$. So $-\frac{\pi}{2} < -\beta < 0$.

$\therefore \quad 0 < \alpha < \frac{\pi}{2}$ and $-\frac{\pi}{2} < -\beta < 0$.

$\therefore \quad -\frac{\pi}{2} < (\alpha - \beta) < \frac{\pi}{2}$

Thus, $(\alpha - \beta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$tan(\alpha - \beta) = \frac{tan\alpha - tan\beta}{1 + tan\alpha\, tan\beta}$$

$\therefore \quad tan(\alpha - \beta) = \frac{x-y}{1+xy} \qquad \alpha - \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \alpha - \beta = tan^{-1}\left(\frac{x-y}{1+xy}\right)$ **$\left(\frac{x-y}{1+xy} \in R \text{ and } x > 0, y > 0; \text{ so } xy \neq -1\right)$**

$\therefore \quad tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Example 8 : Prove :

(1) $tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24} = tan^{-1}\left(\frac{1}{2}\right)$

(2) $cot^{-1}\frac{1}{2} + cot^{-1}\frac{1}{3} = \frac{3\pi}{4}$

(3) $tan^{-1}\frac{1}{7} + tan^{-1}\frac{4}{7} + tan^{-1}\frac{9}{7} = \frac{\pi}{2}$

Solution : (1) L.H.S. $= tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24}$

$= tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right)$ $\quad \left(\frac{2}{11}\times\frac{7}{24}<1\right)$

$= tan^{-1}\left(\frac{48+77}{264-14}\right) = tan^{-1}\left(\frac{125}{250}\right) = tan^{-1}\frac{1}{2}$ = R.H.S.

(2) L.H.S. $= cot^{-1}\frac{1}{2} + cot^{-1}\frac{1}{3}$

$= tan^{-1}2 + tan^{-1}3$ $\quad (2 > 0, 3 > 0)$

$= \pi + tan^{-1}\left(\frac{2+3}{1-2\times3}\right)$ $\quad (2\times3 > 1)$

$= \pi + tan^{-1}(-1)$

$= \pi - tan^{-1}(1)$ $\quad (tan^{-1}(-x) = -tan^{-1}x)$

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ = R.H.S.

(3) L.H.S. $= tan^{-1}\frac{1}{7} + tan^{-1}\frac{4}{7} + tan^{-1}\frac{9}{7}$

$= tan^{-1}\left(\frac{\frac{1}{7}+\frac{4}{7}}{1-\frac{1}{7}\times\frac{4}{7}}\right) + tan^{-1}\frac{9}{7}$ $\quad \left(\frac{1}{7}\times\frac{4}{7} < 1\right)$

$= tan^{-1}\left(\frac{7+28}{49-4}\right) + tan^{-1}\frac{9}{7}$

$= tan^{-1}\left(\frac{35}{45}\right) + tan^{-1}\left(\frac{9}{7}\right)$

$= tan^{-1}\left(\frac{7}{9}\right) + tan^{-1}\left(\frac{9}{7}\right)$

$= \frac{\pi}{2}$ = R.H.S. $\quad \left(\frac{7}{9}\times\frac{9}{7} = 1\right)$

Example 9 : Prove that : $3sin^{-1}x = sin^{-1}(3x - 4x^3)$, if $-\frac{1}{2} \le x \le \frac{1}{2}$.

Solution : Let $sin^{-1}x = \theta$. $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $|x| \le 1$. Then $x = sin\theta$

Now, $sin3\theta = 3sin\theta - 4sin^3\theta$

$\therefore \quad sin3\theta = 3x - 4x^3$

Now, $-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow sin\left(-\frac{\pi}{6}\right) \le sin\theta \le sin\frac{\pi}{6}$

$\Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$ $\quad \left(sin \text{ is } \uparrow \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\Rightarrow -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$

Since $sin3\theta = 3x - 4x^3, -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$

$\therefore \quad 3\theta = sin^{-1}(3x - 4x^3)$

$\therefore \quad 3sin^{-1}x = sin^{-1}(3x - 4x^3)$

Example 10 : Prove : **(1)** $tan^{-1}\sqrt{\frac{a-x}{a+x}} = \frac{1}{2}cos^{-1}\frac{x}{a}, -a < x < a, a \in R^+$

(2) $cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right) = \frac{\pi}{2} - \frac{x}{2}, \frac{\pi}{2} < x < \pi$

(3) $tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x, 0 < x < 1.$

Solution : (1) Consider $tan^{-1}\sqrt{\frac{a-x}{a+x}}, -a < x < a$

$-a < x < a \Rightarrow -1 < \frac{x}{a} < 1 \qquad (a \in R^+)$

$\therefore \quad \frac{x}{a} \in (-1, 1)$

$\therefore \quad \exists\theta \in (0, \pi)$ such that $cos\theta = \frac{x}{a}$ or $\theta = cos^{-1}\frac{x}{a}$

$0 < \theta < \pi$. So, $0 < \frac{\theta}{2} < \frac{\pi}{2}$

Now, $tan^{-1}\sqrt{\frac{a-x}{a+x}} = tan^{-1}\sqrt{\frac{a-acos\theta}{a+acos\theta}}$

$= tan^{-1}\sqrt{\frac{1-cos\theta}{1+cos\theta}}$

$= tan^{-1}\sqrt{tan^2\frac{\theta}{2}}$

$= tan^{-1}\left|tan\frac{\theta}{2}\right|$

$= tan^{-1}\left(tan\frac{\theta}{2}\right) \qquad \left(0 < \frac{\theta}{2} < \frac{\pi}{2}\right)$

$= \frac{\theta}{2} \qquad \left(\frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$= \frac{1}{2}cos^{-1}\frac{x}{a}$

(2) L.H.S. $= cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right), \frac{\pi}{2} < x < \pi$

$= cot^{-1}\left(\frac{\sqrt{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)^2}+\sqrt{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)^2}}{\sqrt{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)^2}-\sqrt{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)^2}}\right)$

$= cot^{-1}\left(\frac{\left|cos\frac{x}{2}+sin\frac{x}{2}\right|+\left|cos\frac{x}{2}-sin\frac{x}{2}\right|}{\left|cos\frac{x}{2}+sin\frac{x}{2}\right|-\left|cos\frac{x}{2}-sin\frac{x}{2}\right|}\right)$

As, $\frac{\pi}{2} < x < \pi \Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

$\therefore$ $cos\frac{x}{2} < sin\frac{x}{2}$ and $cos\frac{x}{2} > 0$, $sin\frac{x}{2} > 0$

$$= cot^{-1}\left(\frac{\left(cos\frac{x}{2} + sin\frac{x}{2}\right) - \left(cos\frac{x}{2} - sin\frac{x}{2}\right)}{\left(cos\frac{x}{2} + sin\frac{x}{2}\right) + \left(cos\frac{x}{2} - sin\frac{x}{2}\right)}\right) \qquad \left(\left|cos\,\frac{x}{2} - sin\,\frac{x}{2}\right| = -\left(cos\,\frac{x}{2} - sin\,\frac{x}{2}\right)\right)$$

$$= cot^{-1}\left(tan\frac{x}{2}\right)$$

$$= cot^{-1}\left(cot\left(\frac{\pi}{2} - \frac{x}{2}\right)\right) \qquad \left(0 < \frac{\pi}{2} - \frac{x}{2} < \frac{\pi}{4}\right)$$

Now, $-\frac{\pi}{4} > -\frac{x}{2} > -\frac{\pi}{2}$. So, $\frac{\pi}{2} - \frac{\pi}{4} > \frac{\pi}{2} - \frac{x}{2} > 0$.

$\therefore$ $0 < \frac{\pi}{2} - \frac{x}{2} < \frac{\pi}{4}$.

$\therefore$ L.H.S. $= \frac{\pi}{2} - \frac{x}{2}$ = R.H.S.

(3) Consider $tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$, $0 < x < 1$

Let $\theta = cos^{-1}x$, $\theta \in [0, \pi]$. $x \in (0, 1)$. Then $x = cos\theta$.

$$\text{L.H.S.} = tan^{-1}\left(\frac{\sqrt{1+cos\,\theta} - \sqrt{1-cos\,\theta}}{\sqrt{1+cos\,\theta} + \sqrt{1-cos\,\theta}}\right)$$

$$= tan^{-1}\left(\frac{\sqrt{2cos^2\frac{\theta}{2}} - \sqrt{2sin^2\frac{\theta}{2}}}{\sqrt{2cos^2\frac{\theta}{2}} + \sqrt{2sin^2\frac{\theta}{2}}}\right)$$

$$= tan^{-1}\left(\frac{\left|cos\frac{\theta}{2}\right| - \left|sin\frac{\theta}{2}\right|}{\left|cos\frac{\theta}{2}\right| + \left|sin\frac{\theta}{2}\right|}\right)$$

As, $0 < x < 1 \Rightarrow 0 < cos\theta < 1$

$\Rightarrow cos\frac{\pi}{2} < cos\,\theta < cos0$

$\Rightarrow 0 < \theta < \frac{\pi}{2}$ (*cos* is ↓ in the 1st quadrant)

$\Rightarrow 0 < \frac{\theta}{2} < \frac{\pi}{4}$

Also, $-\frac{\pi}{4} < -\frac{\theta}{2} < 0$

So, $0 < \left(\frac{\pi}{4} - \frac{\theta}{2}\right) < \frac{\pi}{4}$

$$\text{L.H.S.} = tan^{-1}\left(\frac{cos\frac{\theta}{2} - sin\frac{\theta}{2}}{cos\frac{\theta}{2} + sin\frac{\theta}{2}}\right) \qquad \left(0 < \frac{\theta}{2} < \frac{\pi}{4}\right)$$

$$= tan^{-1}\left(\frac{1 - tan\frac{\theta}{2}}{1 + tan\frac{\theta}{2}}\right) \qquad \left(cos\frac{\theta}{2} \neq 0. \text{ Why ?}\right)$$

$$= tan^{-1}\left(tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)$$

$$= \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{1}{2}cos^{-1}x = \text{R.H.S.} \qquad \left(\frac{\pi}{4} - \frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right)\right)$$

2.15 Inter-relations Between the Inverse Functions

(1) $sin^{-1}x = cos^{-1}\sqrt{1-x^2} = tan^{-1}\frac{x}{\sqrt{1-x^2}}$, **if** $0 < x < 1$.

(2) $cos^{-1}x = sin^{-1}\sqrt{1-x^2} = tan^{-1}\frac{\sqrt{1-x^2}}{x}$, **if** $0 < x < 1$.

(3) $tan^{-1}x = cos^{-1}\frac{1}{\sqrt{1+x^2}} = sin^{-1}\frac{x}{\sqrt{1+x^2}}$, **if** $x > 0$

Proof : Suppose, $sin^{-1}x = \theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \in (0, 1)$. So $sin\theta = x$

Also, $sin\theta = x > 0$. Hence, $\theta \in \left(0, \frac{\pi}{2}\right)$.

Also, $cos^2\theta = 1 - sin^2\theta = 1 - x^2$

$\therefore \quad cos\theta = \sqrt{1-x^2}$ $\quad\quad\left(cos\theta > 0 \text{ in } \left(0, \frac{\pi}{2}\right)\right)$

$\therefore \quad \theta = cos^{-1}\sqrt{1-x^2}$ $\quad\quad\left(\theta \in \left(0, \frac{\pi}{2}\right), 1 < \sqrt{1-x^2} < 1\right)$

$\therefore \quad sin^{-1}x = cos^{-1}\sqrt{1-x^2}$

Also, $tan\theta = \frac{sin\theta}{cos\theta}$

$tan\theta = \frac{x}{\sqrt{1-x^2}}$

$\therefore \quad \theta = tan^{-1}\frac{x}{\sqrt{1-x^2}}$, as $\theta \in \left(0, \frac{\pi}{2}\right)$.

$\therefore \quad sin^{-1}x = tan^{-1}\frac{x}{\sqrt{1-x^2}}$.

Similarly (2) and (3) can be proved.

Example 11 : Prove : $sin^{-1}\frac{3}{5} + cos^{-1}\frac{15}{17} + sin^{-1}\frac{36}{85} = \frac{\pi}{2}$

$$\text{L.H.S.} = sin^{-1}\frac{3}{5} + cos^{-1}\frac{15}{17} + sin^{-1}\frac{36}{85}$$

$$= tan^{-1}\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}} + tan^{-1}\frac{\sqrt{1-\frac{225}{289}}}{\frac{15}{17}} + tan^{-1}\frac{\frac{36}{85}}{\sqrt{1-\frac{36^2}{85^2}}}$$

$$= tan^{-1}\frac{3}{\sqrt{25-9}} + tan^{-1}\frac{\sqrt{289-225}}{15} + tan^{-1}\frac{36}{\sqrt{85^2-36^2}}$$

$$= tan^{-1}\frac{3}{4} + tan^{-1}\frac{8}{15} + tan^{-1}\frac{36}{77}$$

$$= tan^{-1}\left(\frac{\frac{3}{4}+\frac{8}{15}}{1-\frac{3}{4}\times\frac{8}{15}}\right) + tan^{-1}\left(\frac{36}{77}\right) \quad\quad \left(\frac{3}{4} \times \frac{8}{15} < 1\right)$$

$$= tan^{-1}\left(\frac{45+32}{60-24}\right) + tan^{-1}\left(\frac{36}{77}\right)$$

$$= tan^{-1}\left(\frac{77}{36}\right) + tan^{-1}\left(\frac{36}{77}\right)$$

$$= \frac{\pi}{2} = \text{R.H.S.} \quad\quad \left(\frac{77}{36} \times \frac{36}{77} = 1\right)$$

Exercise 2.2

1. Find the value of :

(1) $sin^{-1}\frac{\sqrt{3}}{2} - cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2tan^{-1}(1)$

(2) $3sin^{-1}\frac{1}{2} + 4cos^{-1}\frac{\sqrt{3}}{2} + sec^{-1}1$

(3) $cot^{-1}(1) + 3sin^{-1}\frac{1}{2} - cosec^{-1}(-2) - 3tan^{-1}\frac{1}{\sqrt{3}}$

(4) $5cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - 4tan^{-1}(-\sqrt{3}) + 3sin^{-1}(1)$

(5) $cos\left(sin^{-1}\left(-\frac{4}{5}\right)\right) + sin\left(tan^{-1}\frac{3}{4}\right) + cos\left(cosec^{-1}\frac{5}{3}\right)$

(6) $sin\left(\frac{\pi}{2} - cos^{-1}\frac{3}{7}\right) + cos\left(\frac{3\pi}{2} - sin^{-1}\frac{2}{7}\right) + cot\left(tan^{-1}\frac{7}{6}\right)$

(7) $sin^{-1}\left(sin\frac{5\pi}{6}\right) + cos^{-1}\left(cos\frac{5\pi}{3}\right) + tan^{-1}\left(tan\frac{7\pi}{3}\right)$

2. Prove :

(1) $tan^{-1}\frac{4}{5} + tan^{-1}\frac{2}{3} = tan^{-1}\frac{22}{7}$

(2) $tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{13} = tan^{-1}\frac{2}{9}$

(3) $tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

(4) $tan^{-1}\frac{1}{3} + \frac{1}{2}tan^{-1}\frac{1}{7} = \frac{\pi}{8}$

(5) $tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{3} - tan^{-1}\frac{1}{7} = tan^{-1}\frac{21}{53}$

(6) $tan^{-1}\frac{1}{5} + tan^{-1}\frac{1}{7} + tan^{-1}\frac{1}{3} + tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

3. Prove :

(1) $cos^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13} = tan^{-1}\left(\frac{56}{33}\right)$

(2) $sin^{-1}\frac{3}{5} + cos^{-1}\frac{4}{5} = cot^{-1}\left(\frac{7}{24}\right)$

(3) $2sin^{-1}\frac{5}{13} = cos^{-1}\frac{119}{169}$

(4) $2sin^{-1}\frac{3}{5} + cos^{-1}\frac{24}{25} = \frac{\pi}{2}$

(5) $2cot^{-1}2 + cosec^{-1}\frac{5}{3} = \frac{\pi}{2}$

(6) $sin^{-1}\frac{3}{5} + sin^{-1}\frac{8}{17} + sin^{-1}\frac{36}{85} = \frac{\pi}{2}$

4. Prove :

(1) $2cot^{-1}\frac{1}{3} + tan^{-1}\frac{3}{4} = \pi$

(2) $cot^{-1}1 + tan^{-1}2 + cot^{-1}\frac{1}{3} = \pi$

(3) $cot^{-1}\frac{1}{5} + \frac{1}{2}cot^{-1}\frac{12}{5} = \frac{\pi}{2}$

(4) $sin^{-1}\frac{12}{13} + cos^{-1}\frac{4}{5} + tan^{-1}\frac{63}{16} = \pi$

*

Miscelleneous Examples :

Example 12 : Prove : $cos^{-1} a + cos^{-1} b + cos^{-1} c = \pi \Rightarrow a^2 + b^2 + c^2 + 2abc = 1$, where $a, b, c \in [-1, 1]$.

Solution : Let $cos^{-1}a = \alpha$, $cos^{-1}b = \beta$, $cos^{-1}c = \gamma$ $\quad [\alpha, \beta, \gamma \in [0, \pi]]$

$\therefore \quad a = cos\alpha, \ b = cos\beta, \ c = cos\gamma$

Now, $cos^{-1}a + cos^{-1}b + cos^{-1}c = \pi$

$\therefore \quad \alpha + \beta + \gamma = \pi$

$\therefore \quad \alpha + \beta = \pi - \gamma$

$\therefore \quad cos(\alpha + \beta) = cos(\pi - \gamma)$

$\therefore \quad cos\alpha \, cos\beta - sin\alpha \, sin\beta = -cos\gamma$

$\therefore \quad cos\alpha \, cos\beta + cos\gamma = sin\alpha \, sin\beta$

$\therefore \quad (cos\alpha \, cos\beta + cos\gamma)^2 = sin^2\alpha \, sin^2\beta$

$\therefore \quad (ab + c)^2 = (1 - a^2)(1 - b^2)$

$\therefore \quad a^2b^2 + 2abc + c^2 = 1 - a^2 - b^2 + a^2b^2$

$\therefore \quad a^2 + b^2 + c^2 + 2abc = 1$

Example 13 : Prove that $cosec[tan^{-1}(cos(cot^{-1}(sec(sin^{-1} a))))] = \sqrt{3 - a^2}$, where $0 < a < 1$.

Solution : L.H.S. $= cosec[tan^{-1}(cos(cot^{-1}(sec(sin^{-1} a))))]$

$$= cosec\left[tan^{-1}\left(cos\left(cot^{-1}\left(sec\left(sec^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)\right] \qquad \left(sin^{-1} a = cos^{-1}\sqrt{1-a^2}\right)$$

$$= cosec\left[tan^{-1}\left(cos\left(cot^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right]$$

$$= cosec\left[tan^{-1}\left(cos\left(tan^{-1}\sqrt{1-a^2}\right)\right)\right] \qquad \left(\sqrt{1-a^2} > 0\right)$$

$$= cosec\left[tan^{-1}\left(cos\left(cos^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right)\right] \qquad \left(tan^{-1}x = cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$= cosec\left(tan^{-1}\frac{1}{\sqrt{2-a^2}}\right)$$

$$= cosec\left(sin^{-1}\frac{\frac{1}{\sqrt{2-a^2}}}{\sqrt{1+\frac{1}{2-a^2}}}\right) \qquad \left(tan^{-1}x = sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)$$

$$= cosec\left(sin^{-1}\frac{1}{\sqrt{3-a^2}}\right)$$

$$= cosec\left(cosec^{-1}\sqrt{3-a^2}\right)$$

$$= \sqrt{3-a^2} = \text{R.H.S.}$$

Example 14 : Solve the following equations :

(1) $tan^{-1}\sqrt{3} + 2tan^{-1}x = \frac{5\pi}{6}$ $\qquad$ **(2)** $tan^{-1}2x + 2tan^{-1}x = \frac{\pi}{2}$

Solution : (1) $tan^{-1}\sqrt{3} + 2tan^{-1}x = \frac{5\pi}{6}$

$\therefore \quad \frac{\pi}{3} + 2tan^{-1}x = \frac{5\pi}{6}$

$\therefore \quad 2tan^{-1}x = \frac{5\pi}{6} - \frac{\pi}{3}$

$\therefore \quad 2tan^{-1}x = \frac{\pi}{2}$

$\therefore \quad tan^{-1}x = \frac{\pi}{4}$

$\therefore \quad x = tan\frac{\pi}{4}$

$\therefore \quad x = 1$

Equations involving inverse trigonometric functions can also be solved. However, as the domain and range of such functions are restricted, one must always verify the answer by substituting the solution in the original equation.

Verification : Putting $x = 1$ in the given equation,

L.H.S. $= tan^{-1}\sqrt{3} + 2tan^{-1}x = \frac{\pi}{3} + 2\left(\frac{\pi}{4}\right) = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6} =$ R.H.S.

$\therefore$ The solution set is $\{1\}$.

(2) $tan^{-1}2x + 2tan^{-1}x = \frac{\pi}{2}$

We observe that if $x \geq 1$, then $2tan^{-1}x \geq 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$

that is $tan^{-1}2x \leq 0$ which is not possible. Since $x \geq 1$.

If $x < 0$, L.H.S. < 0, R.H.S. > 0. This is not possible.

$\therefore \quad 0 < x < 1.$

Now, $tan^{-1}2x + 2tan^{-1}x = \frac{\pi}{2}$

$\therefore \quad tan^{-1}2x + tan^{-1}x + tan^{-1}x = \frac{\pi}{2}$

$\therefore \quad tan^{-1}2x + tan^{-1}\left(\frac{x+x}{1-x^2}\right) = \frac{\pi}{2}$ **$(0 < x^2 < 1)$**

$\therefore \quad tan^{-1}2x + tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{2}$

We know that, $xy = 1 \Leftrightarrow tan^{-1}x + tan^{-1}y = \frac{\pi}{2}$

$\therefore \quad 2x \cdot \frac{2x}{1-x^2} = 1$

$\therefore \quad 4x^2 = 1 - x^2$

$\therefore \quad 5x^2 = 1$

$\therefore \quad x^2 = \frac{1}{5}$

$\therefore \quad x = \pm\frac{1}{\sqrt{5}}$. But $x > 0$

$\therefore \quad x = \frac{1}{\sqrt{5}}$

Verification : Taking $x = \frac{1}{\sqrt{5}}$.

L.H.S. $= tan^{-1}\frac{2}{\sqrt{5}} + 2tan^{-1}\frac{1}{\sqrt{5}}$

$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\frac{1}{\sqrt{5}} + tan^{-1}\frac{1}{\sqrt{5}}$

$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\left(\frac{\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{5}}}{1-\frac{1}{5}}\right)$

$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\left(\frac{\sqrt{5}+\sqrt{5}}{5-1}\right)$

$= tan^{-1}\frac{2}{\sqrt{5}} + tan^{-1}\frac{\sqrt{5}}{2}$

$= \frac{\pi}{2}$ = R.H.S.

The solution set is $\left\{\frac{1}{\sqrt{5}}\right\}$.

Example 15 : If $0 < x < 1$ and if $tan^{-1}(1 - x)$, $tan^{-1}x$ and $tan^{-1}(1 - x)$ are in arithmetic progression, prove that $x^3 + x^2 = 1$.

Solution : As $tan^{-1}(1 - x)$, $tan^{-1}x$ and $tan^{-1}(1 - x)$ are in A.P.

$2tan^{-1}x = tan^{-1}(1 - x) + tan^{-1}(1 + x)$

$\therefore \quad tan^{-1}x + tan^{-1}x = tan^{-1}\frac{1-x+1+x}{1-(1-x^2)}$ **$(1 - x > 0,\ 1 + x > 0,\ 0 < 1 - x^2 < 1)$**

$\therefore \quad tan^{-1}\left(\frac{2x}{1-x^2}\right) = tan^{-1}\left(\frac{2}{x^2}\right)$ **$(0 < x^2 < 1)$**

$\therefore \quad \frac{2x}{1-x^2} = \frac{2}{x^2}$ **(tan^{-1} is one-one)**

$\therefore \quad x^3 = 1 - x^2$

$\therefore \quad x^3 + x^2 = 1$

Example 16 : Solve $cos^{-1}x + sin^{-1}2x = \frac{\pi}{6}$

Solution : $cos^{-1}x + sin^{-1}2x = \frac{\pi}{6}$

Let $cos^{-1}x = \alpha$, $\alpha \in [0, \pi]$. Then, $x = cos\alpha$.

$\therefore \quad sin\alpha = \sqrt{1-cos^2\alpha} = \sqrt{1-x^2}$ **($sin\alpha \geq 0$ as $\alpha \in [0, \pi]$)**

Let $sin^{-1}2x = \beta$, $\beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, $2x = sin\beta$.

$\therefore \quad cos\beta = \sqrt{1-4x^2}$ **$\left(cos\beta \geq 0 \text{ as } \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$**

Now, $cos^{-1}x + sin^{-1}2x = \frac{\pi}{6}$

$\therefore \quad \alpha + \beta = \frac{\pi}{6}$

$\therefore \quad sin(\alpha + \beta) = sin\frac{\pi}{6}$

$\therefore \quad sin\alpha\, cos\beta + cos\alpha\, sin\beta = \frac{1}{2}$

$\therefore \quad \sqrt{1-x^2}\ \sqrt{1-4x^2} + x(2x) = \frac{1}{2}$

$\therefore \quad \sqrt{1-x^2}\ \sqrt{1-4x^2} = \frac{1}{2} - 2x^2$

$\therefore \quad \sqrt{1-5x^2+4x^4} = \frac{1}{2} - 2x^2$

$\therefore \quad 1 - 5x^2 + 4x^4 = \left(\frac{1}{2} - 2x^2\right)^2$

$\therefore \quad 1 - 5x^2 + 4x^4 = \frac{1}{4} - 2x^2 + 4x^4$

$\therefore \quad 3x^2 = \frac{3}{4}$

$\therefore \quad x^2 = \frac{1}{4}$

$\therefore \quad x = \pm\frac{1}{2}$

Verification : For $x = \frac{1}{2}$,

L.H.S. $= cos^{-1}\frac{1}{2} + sin^{-1}1 = \frac{\pi}{3} + \frac{\pi}{2} \neq \frac{\pi}{6} \neq$ R.H.S.

For $x = -\frac{1}{2}$,

L.H.S. $= cos^{-1}\left(-\frac{1}{2}\right) + sin^{-1}(-1)$

$= \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6} =$ R.H.S.

$\therefore$ The solution set is $\left\{-\frac{1}{2}\right\}$.

Exercise 2

1. Prove :

(1) $sin^{-1}(2x\sqrt{1-x^2}) = 2sin^{-1}x, \quad |x| < \frac{1}{\sqrt{2}}$

(2) $cos^{-1}(2x^2 - 1) = 2cos^{-1}x, \quad 0 < x < 1$

(3) $cos^{-1}(4x^3 - 3x) = 3cos^{-1}x, \quad \frac{1}{2} < x < 1$

(4) $cot^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\pi}{2} - \frac{1}{2}\,tan^{-1}x$

(5) $sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2tan^{-1}x, \ |x| \leq 1$

(6) $tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = 3tan^{-1}x, \ 0 < x < \frac{1}{\sqrt{3}}$.

(7) $cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right) = \frac{x}{2}, \quad 0 < x < \frac{\pi}{2}$.

(8) $tan^{-1}\left(\frac{\sqrt{1+cosx}+\sqrt{1-cosx}}{\sqrt{1+cosx}-\sqrt{1-cosx}}\right) = \frac{\pi}{4} - \frac{x}{2}, \quad \pi < x < \frac{3\pi}{2}$.

(9) $tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\,cos^{-1}x^2, \quad -1 < x < 1, \quad x \neq 0$

(10) $\tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{a}{b}\right) - x,\ -\frac{\pi}{2} < x < \frac{\pi}{2},\ \frac{a}{b}\tan x > -1$

(11) $\sin^{-1}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right) = \frac{\pi}{4} + x,\quad -\frac{\pi}{4} < x < \frac{\pi}{4}$

2. (1) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then prove that $x + y + z = xyz$

(2) If $\cot^{-1}\frac{1}{x} + \cot^{-1}\frac{1}{y} + \cot^{-1}\frac{1}{z} = \frac{\pi}{2}$, then prove that $xy + yz + zx = 1$

(3) If $\cot^{-1}a + \cot^{-1}b + \cot^{-1}c = \pi$, then prove that $ab + bc + ca = 1$

(4) If $a > b > c > 0$, then prove that $\cot^{-1}\left(\dfrac{ab+1}{a-b}\right) + \cot^{-1}\left(\dfrac{bc+1}{b-c}\right) + \cot^{-1}\left(\dfrac{ca+1}{c-a}\right) = \pi$.

(5) If $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{zx}{yr} + \tan^{-1}\frac{xy}{zr} = \frac{\pi}{2}$, then prove that $x^2 + y^2 + z^2 = r^2$.

(6) If $\tan^{-1}\sqrt{\frac{ar}{bc}} + \tan^{-1}\sqrt{\frac{br}{ca}} + \tan^{-1}\sqrt{\frac{cr}{ab}} = \pi$, then prove that $a + b + c = r$. $(a, b, c, r > 0)$

(7) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, then prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

(8) Prove that $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$

(9) Prove : $\sum_{r=1}^{n} \tan^{-1}\left(\dfrac{1}{1+r(r+1)}\right) = \tan^{-1}(n + 1) - \frac{\pi}{4}$.

(10) $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = \begin{cases} 0, & \text{if } \frac{\pi}{4} < A < \frac{\pi}{2} \\ \pi, & \text{if } 0 < A < \frac{\pi}{4} \end{cases}$

3. Solve the following equations :

(1) $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \frac{\pi}{4}$.

(2) $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$.

(3) $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

(4) $\sin^{-1}x + \cos^{-1}2x = \frac{\pi}{6}$

(5) $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$

(6) $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\frac{8}{31}$

(7) $\tan^{-1}2x + \tan^{-1}\left(\dfrac{1}{x+4}\right) = \frac{\pi}{2}$

4. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A (1 mark)

(1) $\sin\left(3\sin^{-1}\frac{1}{3}\right) = \ldots\ldots$ ☐

(a) $\frac{23}{27}$ (b) $\frac{1}{3}$ (c) $\frac{27}{23}$ (d) $\frac{2\sqrt{3}}{9}$

(2) If $sin^{-1}x = \frac{\pi}{7}$ for some $x \in (-1, 1)$, then the value of $cos^{-1}x = ...$

(a) $\frac{3\pi}{14}$ (b) $\frac{5\pi}{14}$ (c) $\frac{\pi}{14}$ (d) $\frac{6\pi}{7}$

(3) $sec^2(tan^{-1}2) + cosec^2(cot^{-1}3) = ...$.

(a) 15 (b) 6 (c) 13 (d) 25

(4) $cos^{-1}\left(cos\frac{7\pi}{6}\right) = ...$.

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $-\frac{\pi}{6}$ (d) $\frac{7\pi}{6}$

(5) The domain of cos^{-1} is ....

(a) $(-\infty, \infty)$ (b) $[0, 1]$ (c) $[0, \pi]$ (d) $[-1, 1]$

(6) The range of tan^{-1} is ....

(a) $(-\pi, \pi)$ (b) R (c) $(0, \pi)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(7) The value of $cos^{-1}\left(cos\left(-\frac{\pi}{3}\right)\right)$ is ....

(a) $-\frac{\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{2\pi}{3}$

(8) $sin^{-1}\left(cos\frac{\pi}{6}\right)$ is equal to ....

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{2}$

(9) The value of $sin^{-1}\left(sin\frac{5\pi}{3}\right)$ is ....

(a) $-\frac{\pi}{3}$ (b) $\frac{5\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

(10) $cos\left(cos^{-1}\left(-\frac{1}{5}\right) + sin^{-1}\left(-\frac{1}{5}\right)\right)$ is ....

(a) $\frac{4}{9}$ (b) $\frac{1}{3}$ (c) 0 (d) $-\frac{1}{3}$

(11) $cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is ....

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{4\pi}{3}$ (d) $\frac{4\pi}{6}$

(12) $sin^{-1}\left(sin\frac{7\pi}{6}\right)$ is ....

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{-\pi}{6}$ (d) $\frac{7\pi}{6}$

(13) $sin\left\{\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right\}$ is ....

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) 1

(14) Value of $sin\left(cos^{-1}\frac{4}{5}\right)$ is ....

(a) $\frac{1}{2}$ (b) $\frac{3}{5}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

(15) Value of $cos\left(tan^{-1}\frac{4}{3}\right)$ is ....

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{3}{5}$

Section B (2 marks)

(16) $2\tan^{-1}5 + \tan^{-1}\frac{5}{12} = ...$

(a) $\frac{\pi}{4}$ (b) $\frac{2\pi}{3}$ (c) π (d) $\frac{\pi}{2}$

(17) If $\sin^{-1}x + \sin^{-1}x = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y = ...$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) π

(18) If $4\sin^{-1}x + \cos^{-1}x = \pi$, then $x = ...$

(a) $-\frac{1}{4}$ (b) $\frac{1}{4}$ (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

(19) $\sin\left(\tan^{-1}\left(\tan\frac{7\pi}{6}\right)\right) + \cos\left(\cos^{-1}\left(\cos\frac{7\pi}{3}\right)\right) = ...$

(a) -1 (b) 0 (c) 1 (d) $\frac{\sqrt{3}}{2}$

(20) If $\cos(2\sin^{-1}x) = \frac{1}{9}$, then the value of $x = ...$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) 1

(21) The value of $\sin[2\sin^{-1}(\cos A)]$ is ......

(a) $\sin A$ (b) $\cos A$ (c) $\cos 2A$ (d) $\sin 2A$

(22) The value of $\sin\left[3\sin^{-1}\left(\frac{1}{5}\right)\right]$ is ......

(a) $-\frac{3}{5}$ (b) $\frac{79}{12}$ (c) $-\frac{71}{125}$ (d) $\frac{71}{125}$

(23) $\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)$ is ......

(a) $-\frac{5\pi}{8}$ (b) $\frac{3\pi}{8}$ (c) $-\frac{3\pi}{8}$ (d) $\frac{13\pi}{8}$

(24) $\sin^{-1}\left(\sin\frac{32\pi}{7}\right)$ is ......

(a) $\frac{3\pi}{7}$ (b) $\frac{4\pi}{7}$ (c) $\frac{18\pi}{7}$ (d) $\frac{32\pi}{7}$

(25) Value of $\cos\left[\frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{2}\right)\right]$ is ......

(a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (c) $\frac{\sqrt{5}-1}{4}$ (d) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(26) $\tan^{-1}2 + \tan^{-1}3$ is ......

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{3\pi}{2}$

(27) The value of $\sin\left[\tan^{-1}(-\sqrt{3}) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ is = ...

(a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 1

(28) $sin^{-1}\frac{3}{5} + tan^{-1}\frac{1}{7}$ is ......

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $sin^{-1}\frac{4}{5}$

(29) The value of $tan\left(cos^{-1}\frac{3}{4} + sin^{-1}\frac{3}{4} - sec^{-1}3\right)$ is ......

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{2\sqrt{3}}$ (d) $\frac{1}{2\sqrt{2}}$

(30) The value of $sec\left[tan^{-1}\left(\frac{b+a}{b-a}\right) - tan^{-1}\left(\frac{a}{b}\right)\right]$ is ......

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4

Section C (3 marks)

(31) The value of $cot\left[\frac{\pi}{4} - 2cot^{-1}3\right]$ is ......

(a) 3 (b) 7 (c) 9 (d) $\frac{3}{4}$

(32) $tan^{-1}\left(\frac{x}{y}\right) - tan^{-1}\left(\frac{x-y}{x+y}\right)$ = ... $\left(\frac{x}{y} \geq 0\right)$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

(33) If $x = \frac{1}{3}$, then the value of $cos(2cos^{-1}x + sin^{-1}x)$ = ...

(a) $-\sqrt{\frac{8}{9}}$ (b) $-\sqrt{\frac{1}{3}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

(34) $cos^{-1}\left(\frac{x}{5}\right) + cosec^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is ......

(a) 1 (b) 3 (c) 5 (d) 4

(35) The value of $cot\left(cosec^{-1}\frac{5}{3} + tan^{-1}\frac{2}{3}\right)$ is ......

(a) $\frac{3}{17}$ (b) $\frac{4}{17}$ (c) $\frac{5}{17}$ (d) $\frac{6}{17}$

(36) $tan\left(2cos^{-1}\frac{3}{5}\right)$ is ......

(a) $\frac{8}{3}$ (b) $\frac{24}{25}$ (c) $\frac{7}{25}$ (d) $-\frac{24}{7}$

(37) The value of $tan\left[\frac{1}{2}cos^{-1}\frac{\sqrt{5}}{3}\right]$ is ......

(a) $\frac{2+\sqrt{3}}{\sqrt{2}}$ (b) $\frac{3-\sqrt{5}}{2}$ (c) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (d) $\frac{\sqrt{5}+1}{4}$

(38) If $0 < x < 1$, then $tan^{-1}\left(\frac{\sqrt{1-x^2}}{1+x}\right)$ is ......

(a) $\frac{1}{2}sin^{-1}\sqrt{\frac{1-x}{2}}$ (b) $\frac{1}{2}cos^{-1}x$ (c) $\frac{1}{2}cot^{-1}\left(\frac{1-x}{1+x}\right)$ (d) $\frac{1}{2}tan^{-1}\left(\frac{x}{2}\right)$

(39) If $cos(2tan^{-1}x) = \frac{1}{2}$, then the value of x is ......

(a) $\frac{1}{\sqrt{3}}$ (b) $1 - \sqrt{3}$ (c) $1 - \frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

(40) The value of $tan\left\{sin^{-1}\left(\frac{3}{5}\right) + cos^{-1}\left(\frac{5}{13}\right)\right\}$ is ......

(a) $-\frac{24}{5}$ (b) $-\frac{22}{15}$ (c) $-\frac{63}{16}$ (d) $-\frac{47}{12}$

(41) If $sin^{-1}\frac{x}{5} + cosec^{-1}\frac{5}{4} = \frac{\pi}{2}$, then x is ......

(a) 1 (b) 2 (c) 3 (d) 4

(42) $sin^{-1}(cos(sin^{-1}x)) + cos^{-1}(sin(cos^{-1}x))$ is ......

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

Section D (4 marks)

(43) $cot^{-1}\left(\frac{\sqrt{1-sinx}+\sqrt{1+sinx}}{\sqrt{1-sinx}-\sqrt{1+sinx}}\right) = ...$ $\left(0 < x < \frac{\pi}{2}\right)$

(a) $\frac{x}{2}$ (b) $\frac{\pi}{2} - 2x$ (c) $2\pi - x$ (d) $\pi - \frac{x}{2}$

(44) If $sin^{-1}\frac{1}{x} = 2tan^{-1}\frac{1}{7} + cos^{-1}\frac{3}{5}$, then x = ...

(a) $\frac{24}{117}$ (b) $\frac{7}{3}$ (c) $\frac{125}{117}$ (d) $-\frac{117}{44}$

(45) If $\alpha = cos^{-1}\left(\frac{4}{5}\right)$, $\beta = tan^{-1}\left(\frac{2}{3}\right)$, $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, then $\alpha - \beta$ = ...

(a) $sin^{-1}\frac{2}{\sqrt{13}}$ (b) $tan^{-1}\left(\frac{1}{18}\right)$ (c) $cos^{-1}\left(\frac{1}{5\sqrt{3}}\right)$ (d) $sin^{-1}\left(\frac{6}{5\sqrt{13}}\right)$

(46) **Match the following :**

Column (A)	Column (B)
(1) $tan^{-1}\left(\frac{1}{3}\right) + tan^{-1}\left(\frac{1}{4}\right)$	(a) $\frac{\pi}{2}$
(2) $sin^{-1}\left(\frac{3}{5}\right) + sin^{-1}\left(\frac{8}{17}\right) + sin^{-1}\left(\frac{36}{85}\right)$	(b) π
(3) $tan^{-1}(1) + cos^{-1}\left(-\frac{1}{2}\right) + sin^{-1}\left(-\frac{1}{2}\right)$	(c) $tan^{-1}\left(\frac{7}{11}\right)$
(4) $2tan^{-1}(5) + tan^{-1}\left(\frac{5}{12}\right)$	(d) $\frac{3\pi}{4}$

(a) 1 - c, 2- b, 3 - d, 4 - a (b) 1 - c, 2 - a, 3 - d, 4 - b

(c) 1 - c, 2 - a, 3 - b, 4 - d (d) 1 - a, 2 - b, 3 - d, 4 - c

(47) $tan\left(2tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = ...$

(a) $\frac{14}{33}$ (b) $\frac{-7}{17}$ (c) $\frac{17}{7}$ (d) $\frac{24}{25}$

(48) If $sin^{-1}(1 - x) - 2sin^{-1}x = \frac{\pi}{2}$, then x is ......

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

(49) The number of values of x satisfying the equation

$tan^{-1}(x + 1) + tan^{-1}x + tan^{-1}(x - 1) = tan^{-1}3x$ is ......

(a) 2 (b) 3 (c) 4 (d) infinite

(50) If $cot^{-1}x + cot^{-1}y + cot^{-1}z = \frac{\pi}{2}$, then $x + y + z = ...$

(a) $xy + yz + zx$ (b) xyz (c) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ (d) $\frac{xy + yz + zx}{3}$

(51) If $sin^{-1}\left(\frac{2a}{1+a^2}\right) + sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2tan^{-1}x$, then x is ...... $(0 < a, b < 1)$

(a) $\frac{a-b}{1+ab}$ (b) $\frac{a+b}{1-ab}$ (c) $\frac{b}{1-ab}$ (d) $\frac{b}{1+ab}$

*

Summary

We have studied the following points in this chapter :

1. Definition of inverse trigonometric functions.
2. Graphs of inverse trigonometric functions.
3. (1) $sin^{-1}(-x) = -sin^{-1}x, \quad |x| \le 1$

   (2) $cos^{-1}(-x) = \pi - cos^{-1}x, \quad |x| \le 1$

   (3) $tan^{-1}(-x) = -tan^{-1}x, \quad x \in R$

   (4) $cot^{-1}(-x) = \pi - cot^{-1}x, \quad x \in R$

   (5) $cosec^{-1}(-x) = -cosec^{-1}x, \quad |x| \ge 1$

   (6) $sec^{-1}(-x) = \pi - sec^{-1}x, \quad |x| \ge 1$
4. (1) $cosec^{-1}x = sin^{-1}\frac{1}{x}, \quad |x| \ge 1$

   (2) $sec^{-1}x = cos^{-1}\frac{1}{x}, \quad |x| \ge 1$

   (3) $cot^{-1}x = tan^{-1}\frac{1}{x}, \quad x > 0$

   $= \pi + tan^{-1}\frac{1}{x}, \quad x < 0$

5. (1) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad |x| \le 1$

(2) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \quad |x| \ge 1$

(3) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad x \in R$

6. If $x > 0$, $y > 0$, then

(1) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

(2) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy > 1$

(3) $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{2}$, if $xy = 1$

(4) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

7. (1) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, if $0 < x < 1$

(2) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$, if $0 < x < 1$

(3) $\tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

Srinivasa Ramanujan : Adulthood in India

On 14 July 1909, Ramanujan was married to a nine-year old bride, Janaki Ammal. In the branch of Hinduism to which Ramanujan belonged, marriage was a formal engagement that was consummated only after the bride turned 17 or 18, as per the traditional calendar.

After the marriage, Ramanujan developed a hydrocele testis, an abnormal swelling of the tunica vaginalis, an internal membrane in the testicle. The condition could be treated with a routine surgical operation that would release the blocked fluid in the scrotal sac. His family did not have the money for the operation, but in January 1910, a doctor volunteered to do the surgery for free.

After his successful surgery, Ramanujan searched for a job. He stayed at friends' houses while he went door to door around the city of Madras (now Chennai) looking for a clerical position. To make some money, he tutored some students at Presidency College who were preparing for their F.A. exam.

In late 1910, Ramanujan was sick again, possibly as a result of the surgery earlier in the year. He feared for his health, and even told his friend, R. Radakrishna Iyer, to "hand these [my mathematical notebooks] over to Professor Singaravelu Mudaliar [mathematics professor at Pachaiyappa's College] or to the British professor Edward B. Ross, of the Madras Christian College." After Ramanujan recovered and got back his notebooks from Iyer, he took a northbound train from Kumbakonam to Villupuram, a coastal city under French control.