MATRICES 4

It is easier to square the circle than to get round a mathematician.
– Augustus De Morgan

Our notion of symmetry is derived from the human face.
Hence we demand symmetry horizontally and in breadth only not vertically nor in depth.
– Blaise Pascal

4.1 Introduction

If you are asked about your weight in *kg*, you can use a real number such as 55 to answer the question. Again if you are asked for your height in *cm*, your answer is another real number say 135. One way to organise these data is to use an order pair. You can represent your weight and height with the order pair (55, 135). The elements of this order pair indicate the information such as weight and height respectively. If we want to include your age in years say 16, then we have order triplet (55, 135, 16). The elements of this triple indicate the information such as weight, height and age in the sequence for an individual. We can write them in a row, like [55 135 16] or in a column, like $\begin{bmatrix} 55 \\ 135 \\ 16 \end{bmatrix}$.

If the above questions are asked to three or four individuals named Rita, Raman, Rahim and John, then the informations can be collected in the order triples as (55, 135, 16), (58.5, 140, 18), (59, 138, 17) and (60.5, 155, 20) respectively. However, it will be nice if we can combine all these triples together in one set of data. If we consider each triple as one column, then we will have all our data in one arrangement. If we organise them in an array form as :

$$\begin{array}{c} \\ \text{Weight} \\ \text{Height} \\ \text{Age} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Rita} & \text{Raman} & \text{Rahim} & \text{John} \end{array} \\ \begin{bmatrix} 55 & 58.5 & 59 & 60.5 \\ 135 & 140 & 138 & 155 \\ 16 & 18 & 17 & 20 \end{bmatrix} \end{array}$$

If there is a selection of soldiers for the Army wing, then they have to collect above data from so many individuals. If the data so collected can be arranged in the precise form as shown above, then it is easy to interpret them. Also, it is easy to make selection of individuals.

The above arrangement of real numbers in a rectangular array is known as a **Matrix** (Plural is matrices). The real numbers are the elements or entries of the matrix.

Matrix is a latin word. The origin of matrices lie with the study of systems of simultaneous linear equations. An important Chinese Text between 300 BC and 200 AD, nine chapters of Mathematical art ***(Chiu Chang Suan Shu)***, give the use of matrix methods to solve simultaneous equations. **Carl Friedrich Gauss** (1777-1855) also gave the method to solve simultaneous linear equations by matrix method.

Matrix operations are used in electronic physics. They are used in computers, budgeting, cost estimation, analysis and experiments. They are also used in cryptography, modern psychology, genetics, industrial management etc.

4.2 Matrix

Any rectangular arrangement or an array of numbers enclosed in brackets such as [ ] or ( ) is called a matrix. We shall consider only real matrices, i.e. elements or entries of the matrices will be real numbers only.

The matrix, $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ has two rows and three columns. So we say that it is a 2×3 matrix. 2×3 is also known the **order** of the matrix.

In general, an $m \times n$ matrix is a matrix having m rows and n columns. It can be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

Here 'a_{ij}' is the element of the matrix in '*ith*' row and '*jth*' column. In a compact form, we can write this matrix as $[a_{ij}]_{m \times n}$. If there is no confusion, we write it as $[a_{ij}]$ also. We denote matrices by A, B, C etc. In the notation of the order $m \times n$ of a matrix, m denotes the number of rows of the matrix and n denotes the number of columns of the matrix. An $m \times n$ matrix is called a rectangular matrix.

Example 1 : Construct 4×3 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = i - j$

Solution : We have matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

Here $a_{ij} = i - j$, so we have $a_{11} = 1 - 1 = 0$, $a_{12} = 1 - 2 = -1$, $a_{13} = 1 - 3 = -2$,

$a_{21} = 2 - 1 = 1$ etc. Thus, we have $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

Difference Between a Determinant and a Matrix :

(1) A determinant has a real value where as a matrix has no real value as it is an arragement of real numbers only.

(2) In a determinant, number of rows is equal to the number of columns where as in a matrix, number of rows may or may not be equal to the number of columns.

Equality of Matrices :

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal if they have the same order and $a_{ij} = b_{ij}$, for all i and j. We denote equal matrices A and B as A = B.

Here, $A = B \Leftrightarrow [a_{ij}]_{m \times n} = [b_{ij}]_{m \times n} \Leftrightarrow a_{ij} = b_{ij} \ \forall \ i = 1, 2, 3, ..., m; \ j = 1, 2, 3, ..., n$

Example 2 : Find x and y, if $\begin{bmatrix} x-1 & 2y \\ x+y & 4 \end{bmatrix} = \begin{bmatrix} 3x-7 & y^2-3 \\ 6 & 4 \end{bmatrix}$.

Solution : Corresponding elements of two matrices must be equal.

$\therefore \quad x - 1 = 3x - 7, \quad 2y = y^2 - 3$ and $x + y = 6$ and $4 = 4$.

$\therefore \quad 2x = 6, \qquad y^2 - 2y - 3 = 0$

$\therefore \quad x = 3, \qquad (y - 3)(y + 1) = 0$

$y = 3$ or $y = -1$

Here, $x = 3$ and $y = 3$ satisfy the equation $x + y = 6$ and $x = 3$, $y = -1$ do not satisfy $x + y = 6$. Hence, $x = 3$ and $y = 3$.

Types of Matrices :

Row Matrix : A $1 \times n$ matrix $[a_{11} \quad a_{12} \quad a_{13} \dots a_{1n}]$ is called a row matrix.

A row matrix has only one row (and any number of columns).

e.g. $A = [3 \quad 5 \quad -1 \quad 4 \quad 0]$ is a 1×5 row matrix.

Column Matrix : An $m \times 1$ matrix $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$ is called a column matrix.

A column matrix has only one column (and any number of rows).

e.g. $A = \begin{bmatrix} 15 \\ 7 \\ 10 \\ -8 \end{bmatrix}$ is a 4×1 column matrix.

Square Matrix : An $n \times n$ matrix is called a square matrix.

A square matrix has the number of columns equal to the number of rows.

For instance $\begin{bmatrix} 5 & -1 & 3 \\ 11 & 2 & 9 \\ -4 & 0 & -7 \end{bmatrix}$ is 3×3 square matrix.

(**Note :** $[a_{ij}]_{1 \times 1}$ matrix is a row matrix, is a column matrix and a square matrix also.)

Diagonal Matrix : If in a square matrix $A = [a_{ij}]_{n \times n}$, we have $a_{ij} = 0$ whenever $i \neq j$, then A is called a diagonal matrix. This is a square matrix in which all entries are zero except possibly those on the diagonal from top left corner to bottom right corner (principal diagonal).

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$ **is a diagonal matrix.**

A diagonal matrix is also denoted as $diag\,[a_{11} \quad a_{22} \quad a_{33} \dots a_{nn}]$.

e.g. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix, i.e. $diag\,[5 \ 0 \ 3]$.

Here, 5, 0, 3 are the elements of the principal diagonal of the matrix A.

Zero Matrix : If all elements of a matrix are zero, then that matrix is known as zero matrix. We denote zero matrix by $[0]_{m \times n}$ or $O_{m \times n}$. $O_{m \times n}$ is also written as O.

Thus, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix. It is a $O_{2 \times 3}$ zero matrix.

4.3 Operations on Matrices

Sum of Two Matrices : If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then their sum is defined as $A + B = [a_{ij} + b_{ij}]_{m \times n}$, i.e. a matrix obtained by taking sum of the corresponding elements of A and B.

For the sum of two matrices, they must have the same number of rows and the same number of columns, otherwise it is not possible to add the matrices. If A and B are both $m \times n$ they are called compatible for sum. In notation $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$.

For instant, if $A = \begin{bmatrix} 1 & 5 \\ 2 & -3 \\ 4 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 1 & 2 \\ -5 & -4 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1-3 & 5+2 \\ 2+1 & -3+2 \\ 4-5 & -7-4 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 3 & -1 \\ -1 & -11 \end{bmatrix}$.

Properties of Matrix Addition :

(1) Commutative Law for Addition :

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then $A + B = B + A$.

Now, $A + B = [a_{ij}] + [b_{ij}]$

$= [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (Commutativity of addition in R)

$= [b_{ij}] + [a_{ij}]$

$= B + A$

$\therefore \quad A + B = B + A$

(2) Associative Law for Addition :

For $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$,

$(A + B) + C = A + (B + C)$.

Now, $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}]$

$= [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij} + (b_{ij} + c_{ij})]$ (Associative law of addition in R)

$= [a_{ij}] + [b_{ij} + c_{ij}]$

$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= A + (B + C)$

$\therefore \quad (A + B) + C = A + (B + C)$

(3) The Identity for Addition of Matrices :

Let $A = [a_{ij}]_{m \times n}$ and $O = [0]_{m \times n}$ be the zero matrix. Then $A + O = O + A = A$

$A + O = [a_{ij}] + [0]$

$= [a_{ij} + 0]$

$= [a_{ij}] = A$ (0 is the additive identity in R)

$\therefore \quad A + O = [a_{ij}]$

By commutative law A + O = O + A

$\therefore$ A + O = O + A = A

Thus, **O is the identity matrix for addition.**

(4) Existence of Additive Inverse :

Let $A = [a_{ij}]_{m \times n}$ be any matrix. Then we have another matrix $[-a_{ij}]_{m \times n}$, so that $A + [-a_{ij}] = O_{m \times n}$.

$$\begin{aligned} A + [-a_{ij}] &= [a_{ij}] + [-a_{ij}] \\ &= [a_{ij} - a_{ij}] \\ &= [\,0\,] \\ &= O_{m \times n} \end{aligned}$$

We denote $[-a_{ij}]$ as $-A$.

By commutative law $A + (-A) = O = (-A) + A$.

Thus, **$-A = [-a_{ij}]$ is called the additive inverse of $A = [a_{ij}]$.**

Difference of Matrices : **If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then the difference of A and B is defined as $A - B = A + (-B) = [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}]$.**

Example 3 : If $A = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 2 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix}$, then find A + B and A − B.

Solution : $A + B = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 5 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2+5 & -3+4 & 4-2 \\ 5+3 & 2+1 & 8+2 \end{bmatrix} = \begin{bmatrix} 7 & 1 & 2 \\ 8 & 3 & 10 \end{bmatrix}$$

$$A - B = A + (-B) = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 2 & 8 \end{bmatrix} + \begin{bmatrix} -5 & -4 & 2 \\ -3 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-5 & -3-4 & 4+2 \\ 5-3 & 2-1 & 8-2 \end{bmatrix} = \begin{bmatrix} -3 & -7 & 6 \\ 2 & 1 & 6 \end{bmatrix}$$

Example 4 : Can we add $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$? Give reason.

Solution : Here A is a 3×2 matrix and B is a 2×2 matrix. They do not have same number of rows. They are not compatible for addition. So we cannot add A and B.

Product of a Matrix with a Scalar and Properties :

If $A = [a_{ij}]$ is an $m \times n$ matrix and k is any real number, then the matrix $[ka_{ij}]$ is called the product of the matrix A by the scalar k. It is denoted by kA. Thus, for $A = [a_{ij}]$, $kA = [ka_{ij}]$.

In kA every element of A gets multiplied by k. (Compare corresponding result for a determinant !)

Properties of Addition of Matrices and of Multiplication of a Matrix by a Scalar :

Suppose, $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices and $k, l \in R$, then

(1) $k(A + B) = kA + kB$ (2) $(k + l)A = kA + lA$ (3) $(kl)A = k(lA)$

(4) $1A = A$ (5) $(-1)A = -A$

Proof : (1) $k(A + B) = k[a_{ij} + b_{ij}]$

$= [k(a_{ij} + b_{ij})]$

$= [ka_{ij} + kb_{ij}]$

$= [ka_{ij}] + [kb_{ij}]$

$= k[a_{ij}] + k[b_{ij}]$

$= kA + kB$

(2) $(k + l)A = (k + l)[a_{ij}]$

$= [(k + l)\, a_{ij}]$

$= [ka_{ij} + la_{ij}]$

$= [ka_{ij}] + [la_{ij}]$

$= k[a_{ij}] + l[a_{ij}]$

$= kA + lA$

(3) $(kl)A = (kl)[a_{ij}]$

$= [(kl)\, a_{ij}]$

$= [k(la_{ij})]$

$= k[la_{ij}]$

$= k[l(a_{ij})]$

$= k(lA)$

(4) $1A = [1 \cdot a_{ij}]$

$= [a_{ij}]$

$= A$

(5) $(-1)A = (-1)[a_{ij}] = [(-1)a_{ij}] = [-a_{ij}] = -A$

Thus, $(-1)\,A = -A$

Example 5 : If $A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ -3 & 1 & -5 & 7 \\ 2 & -9 & -8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 4 & 0 & 1 & -6 \\ -2 & 3 & 6 & -7 \end{bmatrix}$, then obtain $3A - 2B$.

Proof : $3A - 2B = 3A + (-2)B$

$$= 3\begin{bmatrix} 4 & 2 & 1 & 0 \\ -3 & 1 & -5 & 7 \\ 2 & -9 & -8 & 5 \end{bmatrix} + (-2)\begin{bmatrix} 1 & 2 & -3 & 5 \\ 4 & 0 & 1 & -6 \\ -2 & 3 & 6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 6 & 3 & 0 \\ -9 & 3 & -15 & 21 \\ 6 & -27 & -24 & 15 \end{bmatrix} + \begin{bmatrix} -2 & -4 & 6 & -10 \\ -8 & 0 & -2 & 12 \\ 4 & -6 & -12 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 12-2 & 6-4 & 3+6 & 0-10 \\ -9-8 & 3+0 & -15-2 & 21+12 \\ 6+4 & -27-6 & -24-12 & 15+14 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 2 & 9 & -10 \\ -17 & 3 & -17 & 33 \\ 10 & -33 & -36 & 29 \end{bmatrix}$$

Example 6 : If $A = \begin{bmatrix} 5 & 4 \\ 0 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 6 & -5 \end{bmatrix}$, then find the matrix X, such that $3A + 2X = 4B$.

Solution : We wish to find matrix X such that $3A + 2X = 4B$

$\therefore$ $(-3A) + (3A + 2X) = (-3A) + 4B$ **(adding additive inverse of 3A)**

$\therefore$ $(-3A + 3A) + 2X = (-3A) + 4B$

$\therefore$ $O + 2X = 4B - 3A$

$\therefore$ $2X = 4B - 3A$ **(O is the identity for addition)**

$\therefore \quad X = \frac{1}{2}(4B - 3A)$

$$\therefore \quad X = \frac{1}{2}\left(4\begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 6 & -5 \end{bmatrix} + (-3)\begin{bmatrix} 5 & 4 \\ 0 & -2 \\ 3 & 6 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 4 & 8 \\ 12 & -16 \\ 24 & -20 \end{bmatrix} + \begin{bmatrix} -15 & -12 \\ 0 & 6 \\ -9 & -18 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4-15 & 8-12 \\ 12+0 & -16+6 \\ 24-9 & -20-18 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -11 & -4 \\ 12 & -10 \\ 15 & -38 \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} & -2 \\ 6 & -5 \\ \frac{15}{2} & -19 \end{bmatrix}$$

Transpose of a Matrix and its Properties :

Transpose of a Matrix : If all the rows of matrix $A = [a_{ij}]_{m \times n}$ are converted into corresponding columns, the matrix so obtained is called the tranpose of A.

If $A = [a_{ij}]_{m \times n}$ is a matrix, then its transpose is $[a_{ji}]_{n \times m}$ is denoted by A^T or A'.

If $A = [a_{ij}]_{m \times n}$, then $A^T = [a_{ji}]_{n \times m}$.

For example, if $A = \begin{bmatrix} 3 & \sqrt{5} & 2 \\ \sqrt{2} & -1 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{5} & -1 \\ 2 & 0 \end{bmatrix}$

Symmetric Matrix : For a square matrix A, if $A^T = A$, then A is called a symmetric matrix. If $A = [a_{ij}]_{n \times n}$, then $A^T = [a_{ji}]_{n \times n}$. As $A^T = A$, so $a_{ij} = a_{ji}$ for all i and j.

Thus, if $A = \begin{bmatrix} 1 & 3 & -5 \\ 3 & 0 & 2 \\ -5 & 2 & -7 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 3 & -5 \\ 3 & 0 & 2 \\ -5 & 2 & -7 \end{bmatrix}$.

We have $A^T = A$, so A is a symmetric matrix.

Skew-Symmetric Matrix : For a square matrix A, if $A^T = -A$, then A is called a skew-symmetric matrix. In such a matrix $A^T = [a_{ji}]_{n \times n}$, $a_{ji} = -a_{ij}$ for all i and j.

Now, when $i = j$, then we have $a_{ii} = -a_{ii}$, for all i.

$\therefore \quad 2a_{ii} = 0$

$\therefore \quad a_{ii} = 0, \forall i.$

This means that all elements on the principal diagonal of a skew-symmetric matrix are zero. Here, $a_{11} = a_{22} = ... = a_{nn} = 0$.

For example, the matrix $A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$, then

$$A^T = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix} = -1\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} = (-1)A = -A$$

$\therefore$ A is a skew-symmetric matrix.

Some properties of Addition and Multiplication Regarding Transpose of Matrix :

(1) $(A + B)^T = A^T + B^T$, (2) $(A^T)^T = A$, (3) $(kA)^T = kA^T$, $k \in R$

Proof : (1) For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$,

$A^T = [a_{ji}]$ and $B^T = [b_{ji}]$, are $n \times m$ matrices.

Now, $A + B = [a_{ij} + b_{ij}] = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$

$\therefore$ $(A + B)^T = [c_{ji}]$

$= [a_{ji} + b_{ji}]$

$= [a_{ji}] + [b_{ji}]$

$\therefore$ $(A + B)^T = A^T + B^T$

(2) Let $A = [a_{ij}]$

$\therefore$ $A^T = [a_{ji}]$ and hence $(A^T)^T = [a_{ij}] = A$

$\therefore$ $(A^T)^T = A$

(3) Suppose $A = [a_{ij}]$

$\therefore$ $kA = [ka_{ij}] = [c_{ij}]$ where $c_{ij} = ka_{ij}$.

$\therefore$ $(kA)^T = [c_{ji}]$

$= [ka_{ji}]$

$= k[a_{ji}]$

$= kA^T$

Example 7 : If $A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix}$, obtain $A + A^T$ and $A - A^T$.

What can you say about the matrices $A + A^T$ and $A - A^T$?

Solution : $A = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix}$. Hence $A^T = \begin{bmatrix} 2 & 3 & -6 \\ -1 & 2 & 3 \\ 5 & -4 & 8 \end{bmatrix}$

Now $A + A^T = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -6 \\ -1 & 2 & 3 \\ 5 & -4 & 8 \end{bmatrix}$

$= \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & -1 \\ -1 & -1 & 16 \end{bmatrix}$

If $B = A + A^T$

Then $B^T = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & -1 \\ -1 & -1 & 16 \end{bmatrix} = B$

Thus, $(A + A^T)^T = A + A^T$. Hence $A + A^T$ is a symmetric matrix.

Again, $A - A^T = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 2 & -4 \\ -6 & 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 3 & -6 \\ -1 & 2 & 3 \\ 5 & -4 & 8 \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 & 11 \\ 4 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$

Let $C = A - A^T$

$\therefore \quad C^T = \begin{bmatrix} 0 & 4 & -11 \\ -4 & 0 & 7 \\ 11 & -7 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -4 & 11 \\ 4 & 0 & -7 \\ -11 & 7 & 0 \end{bmatrix}$

$\therefore \quad C^T = -C$

$\therefore \quad (A - A^T)^T = -(A - A^T)$. Hence $A - A^T$ is a skew-symmetric matrix.

Example 8 : Simplify $cosec\theta \begin{bmatrix} cosec\theta & -cot\theta \\ cot\theta & -cosec\theta \end{bmatrix} + cot\theta \begin{bmatrix} -cot\theta & cosec\theta \\ -cosec\theta & cot\theta \end{bmatrix}$

Solution : $cosec\theta \begin{bmatrix} cosec\theta & -cot\theta \\ cot\theta & -cosec\theta \end{bmatrix} + cot\theta \begin{bmatrix} -cot\theta & cosec\theta \\ -cosec\theta & cot\theta \end{bmatrix}$

$= \begin{bmatrix} cosec^2\theta & -cosec\theta\, cot\theta \\ cosec\theta\, cot\theta & -cosec^2\theta \end{bmatrix} + \begin{bmatrix} -cot^2\theta & cot\theta\, cosec\theta \\ -cot\theta\, cosec\theta & cot^2\theta \end{bmatrix}$

$= \begin{bmatrix} cosec^2\theta - cot^2\theta & -cosec\theta\, cot\theta + cot\theta\, cosec\theta \\ cot\theta\, cosec\theta - cot\theta\, cosec\theta & -cosec^2\theta + cot^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Example 9 : Prove that if A is a square matrix, then $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew-symmetric matrix and every matrix A can be uniquely written as a sum $A = B + C$ where B is a symmetric matrix and C is a skew-symmetric matrix.

Solution : If $B = A + A^T$, then $B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B$

$\therefore \quad B = A + A^T$ is a symmetric matrix.

Let $C = A - A^T$

Then $C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$

$\therefore \quad C = A - A^T$ is a skew-symmetric matrix.

Also $A = \frac{1}{2}(A + A^T + A - A^T) = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}B + \frac{1}{2}C.$

$\therefore$ A is a sum of a symmetric matrix and a skew-symmetric matrix as $\frac{1}{2}$B and $\frac{1}{2}$C are symmetric and skew symmetric matrices respectively.

Conversely let A = B + C where B is a symmetric matrix and C is a skew-symmetric matrix.

$\therefore$ $B^T = B$ and $C^T = -C$

Now $A^T = B^T + C^T = B - C$

$\therefore$ $A + A^T = 2B \quad A - A^T = 2C$

$\therefore$ $B = \frac{A+A^T}{2}, \quad C = \frac{A-A^T}{2}$

$\therefore$ The expression for A as a sum of a symmetric matrix and a skew-symmetric matrix is unique.

Exercise 4.1

1. If $A = \begin{bmatrix} 2 & -4 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 0 & 5 \\ 4 & -2 \end{bmatrix}$, then find A + B, A − B, 2A + B, A − 2B.

2. If $A = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$, then obtain $A + A^T$ and $A - A^T$.

3. If A = *diag* [1 −1 2] and B = *diag* [3 2 1], find B − A, 2A + 3B.

4. Solve the matrix equation $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 4\begin{bmatrix} 2x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \end{bmatrix}$.

5. If $a_{ij} = \frac{(i-2j)^2}{3}$, obtain $[a_{ij}]_{2 \times 2}$.

6. If $A = \begin{bmatrix} 1 & 2 & 5 \\ 5 & 1 & 1 \\ 3 & 0 & 4 \end{bmatrix}$, find $A - 2A^T$.

7. If $\begin{bmatrix} x+y & xy \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ -8 & 3 \end{bmatrix}$, then find x and y.

8. Obtain a, b, c, d, if $\begin{bmatrix} a-2b & c+d \\ 2a-b & 3a-c \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 7 & 10 \end{bmatrix}$.

9. Find matrix A and B, if $A + B = \begin{bmatrix} 2 & 5 \\ 9 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 6 & 3 \\ -1 & 0 \end{bmatrix}$.

10. Find matrix X, if 5A − 3X = 2B, where $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$.

11. Suppose $A = \begin{bmatrix} 3 & 1 & 1 \\ -12 & -3 & 0 \\ -9 & -1 & -12 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 2 & -4 \\ 5 & 1 & 9 \end{bmatrix}$ and 3A + 4B − X = O, then find matrix X.

12. Find a and b, if $2\begin{bmatrix} 5 & a \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 7 & 0 \end{bmatrix}$.

*

Multiplication of Matrices :

The product AB of two matrices A and B is defined only if the number of columns of A is equal to the number of rows of B.

Suppose, $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices. Then their product $AB = [c_{ij}]_{m \times p}$ is defined by, $c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{i3} \cdot b_{3j} + ... + a_{in} \cdot b_{nj}$.

To obtain the entry in i th row and jth column of matrix AB, we multiply elements of the ith row of the matrix A with corresponding elements of the jth column of the matrix B and then we take the sum of all these products. Thus, for $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, the product $AB = \left[\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right]_{m \times p}$.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ we say A and B are compatible for multiplication.

Example 10 : If $A = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, find AB and BA and also show that $AB \neq BA$.

Solution :
$$AB = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2(1)+3(3) & 2(-2)+3(4) \\ -4(1)+5(3) & -4(-2)+5(4) \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 8 \\ 11 & 28 \end{bmatrix} \qquad \text{(i)}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1(2)+(-2)(-4) & 1(3)+(-2)5 \\ 3(2)+4(-4) & 3(3)+4(5) \end{bmatrix}$$
$$= \begin{bmatrix} 10 & -7 \\ -10 & 29 \end{bmatrix} \qquad \text{(ii)}$$

Observing results (i) and (ii), we can say that $AB \neq BA$.

Example 11 : If $A = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & 4 \\ 0 & 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 4 & -2 \\ 2 & -3 \end{bmatrix}$, then find AB. Is BA defined ? Why ?

Solution :
$$AB = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 2 & 4 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & -2 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+(-1)4+1(2) & 2(1)+(-1)(-2)+1(-3) \\ -3(1)+2(4)+4(2) & -3(1)+2(-2)+4(-3) \\ 0(1)+3(4)+(-5)2 & 0(1)+3(-2)+(-5)(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 13 & -19 \\ 2 & 9 \end{bmatrix}$$

BA is not defined because, B has two columns and A has three rows.

Example 12 : If $A = \begin{bmatrix} cos^2\alpha & cos\alpha\, sin\alpha \\ cos\alpha\, sin\alpha & sin^2\alpha \end{bmatrix}$, $B = \begin{bmatrix} cos^2\beta & cos\beta\, sin\beta \\ cos\beta\, sin\beta & sin^2\beta \end{bmatrix}$ and

$\alpha - \beta = (2n - 1)\frac{\pi}{2}$, $n \in Z$, then prove that AB is zero matrix.

Solution : $AB = \begin{bmatrix} cos^2\alpha & cos\alpha\, sin\alpha \\ cos\alpha\, sin\alpha & sin^2\alpha \end{bmatrix} \begin{bmatrix} cos^2\beta & cos\beta\, sin\beta \\ cos\beta\, sin\beta & sin^2\beta \end{bmatrix}$

$$= \begin{bmatrix} cos^2\alpha\, cos^2\beta + cos\alpha\, sin\alpha\, cos\beta\, sin\beta & cos^2\alpha\, cos\beta\, sin\beta + cos\alpha\, sin\alpha\, sin^2\beta \\ cos\alpha\, sin\alpha\, cos^2\beta + sin^2\alpha\, cos\beta\, sin\beta & cos\alpha\, sin\alpha\, cos\beta\, sin\beta + sin^2\alpha\, sin^2\beta \end{bmatrix}$$

$$= \begin{bmatrix} cos\,\alpha\, cos\,\beta\,(cos\,\alpha\, cos\,\beta + sin\,\alpha\, sin\,\beta) & cos\,\alpha\, sin\,\beta\,(cos\,\alpha\, cos\,\beta + sin\,\alpha\, sin\,\beta) \\ cos\,\beta\, sin\,\alpha\,(cos\,\alpha\, cos\,\beta + sin\,\alpha\, sin\,\beta) & sin\,\alpha\, sin\,\beta\,(cos\,\alpha\, cos\,\beta + sin\,\alpha\, sin\,\beta) \end{bmatrix}$$

$$= \begin{bmatrix} cos\,\alpha\, cos\,\beta\, cos\,(\alpha-\beta) & cos\,\alpha\, sin\,\beta\, cos\,(\alpha-\beta) \\ cos\,\beta\, sin\,\alpha\, cos\,(\alpha-\beta) & sin\,\alpha\, sin\,\beta\, cos\,(\alpha-\beta) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \left(cos(\alpha - \beta) = cos(2n-1)\frac{\pi}{2} = 0\right)$$

Example 13 : If $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix}$, prove that $(A + B)^2 \neq A^2 + 2AB + B^2$.

Solution : We have, $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & -3-12 \\ 2+8 & -6+16 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -15 \\ 10 & 10 \end{bmatrix}$$

$$\therefore \quad B^2 = BB = \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+20 & -4-8 \\ -5-10 & 20+4 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & -12 \\ -15 & 24 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} -1-15 & 4+6 \\ -2+20 & 8-8 \end{bmatrix} = \begin{bmatrix} -16 & 10 \\ 18 & 0 \end{bmatrix}$$

$$\therefore \quad 2AB = \begin{bmatrix} -32 & 20 \\ 36 & 0 \end{bmatrix}$$

$$\therefore \quad A^2 + 2AB + B^2 = \begin{bmatrix} -5 & -15 \\ 10 & 10 \end{bmatrix} + \begin{bmatrix} -32 & 20 \\ 36 & 0 \end{bmatrix} + \begin{bmatrix} 21 & -12 \\ -15 & 24 \end{bmatrix}$$

$$\therefore \quad A^2 + 2AB + B^2 = \begin{bmatrix} -16 & -7 \\ 31 & 34 \end{bmatrix} \qquad \text{(i)}$$

$$\therefore \quad A + B = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad (A + B)^2 = (A + B)(A + B)$$

$$= \begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 0+7 & 0+2 \\ 0+14 & 7+4 \end{bmatrix}$$

$$\therefore \quad (A + B)^2 = \begin{bmatrix} 7 & 2 \\ 14 & 11 \end{bmatrix} \qquad \text{(ii)}$$

From (i) and (ii), we can see that $(A + B)^2 \neq A^2 + 2AB + B^2$.

[**Note** : For the matrix A, $A^2 = AA$ and we do not take simply squares of entries of A.]

Properties of Matrix Multiplication :

Matrix multiplication has the following properties. We shall assume them without proof.

(1) Distributive Laws :

(i) For $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$, $C = [c_{ij}]_{n \times p}$

$A(B + C) = AB + AC$

(ii) For matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, $C = [c_{ij}]_{n \times p}$

$(A + B)C = AC + BC$

(2) Associative Laws :

(ii) For matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$, $C = [c_{ij}]_{p \times q}$

$A(BC) = (AB)C$

Identity Matrix (Unit Matrix) : A square matrix in which all elements on principal diagonal are 1 and the rest of them are 0 is called an identity or a unit matrix. Identity matrix is denoted by I.

Thus, $I = [a_{ij}]_{n \times n}$ where $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

I is also represented as I_n or $I_{n \times n}$.

i.e. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix.

As this identity matrix is a 3×3 matrix, it is denoted by $I_{3 \times 3}$ or simply by I_3.

If $A = [a_{ij}]_{n \times n}$, then for the identity matrix I_n we have $AI_n = I_nA = A$.

(**Note :** A symbol δ_{ij} called Kronecker delta is used to define I.

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Thus, $I = [\delta_{ij}]$)

Scalar Matrix : If $k \in R$, then kI_n is called a scalar matrix.

Thus, $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a scalar matrix.

Here $k = 4$ and $A = 4I_3$.

Example 14 : If $A = [x \quad y \quad z]$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then find (AB)C.

Solution : Now, $AB = [x \quad y \quad z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$$= [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz]$$

$$\therefore \quad (AB)C = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [(ax + hy + gz)x + (hx + by + fz)y + (gx + fy + cz)z]$$

$$= [ax^2 + hxy + gzx + hxy + by^2 + fzy + gxz + fyz + cz^2]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$$

Example 15 : If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ 6 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, find a 2×2 matrix X such that $BX - AC = O$

Solution : Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $BX - AC = O$

$$\therefore \quad \begin{bmatrix} -2 & 5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -2a + 5c & -2b + 5d \\ 6a + c & 6b + d \end{bmatrix} - \begin{bmatrix} -9 & -6 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -2a + 5c + 9 & -2b + 5d + 6 \\ 6a + c - 43 & 6b + d - 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \quad -2a + 5c + 9 = 0,$

$6a + c - 43 = 0$

$\therefore \quad -6a + 15c = -27, \quad$ (i)

$6a + c = 43 \quad$ (ii)

$\therefore$ Adding (i) and (ii),

$16c = 16 \Rightarrow c = 1$ and $a = 7$

$-2b + 5d + 6 = 0$

$6b + d - 22 = 0$

$-6b + 15d = -18 \quad$ (iii)

$6b + d = 22 \quad$ (iv)

$\therefore$ Adding (iii) and (iv),

$16d = 4 \Rightarrow d = \frac{1}{4}$ and $b = \frac{29}{8}$

Hence $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 7 & \frac{29}{8} \\ 1 & \frac{1}{4} \end{bmatrix}$.

Example 16 : Prove that if $A(x) = \begin{bmatrix} cos x & -sin x \\ sin x & cos x \end{bmatrix}$, then $A(\alpha) A(\beta) = A(\alpha + \beta)$ and deduce that $A(\alpha) A(\beta)$ is the identity matrix I_2, where $\alpha + \beta = 2n\pi$, $n \in Z$.

Solution : $A(\alpha) A(\beta) = \begin{bmatrix} cos\,\alpha & -sin\,\alpha \\ sin\,\alpha & cos\,\alpha \end{bmatrix} \begin{bmatrix} cos\,\beta & -sin\,\beta \\ sin\,\beta & cos\,\beta \end{bmatrix}$

$= \begin{bmatrix} cos\,\alpha\, cos\,\beta - sin\,\alpha\, sin\,\beta & -cos\,\alpha\, sin\,\beta - sin\,\alpha\, cos\,\beta \\ sin\,\alpha\, cos\,\beta + cos\,\alpha\, sin\,\beta & -sin\,\alpha\, sin\,\beta + cos\,\alpha\, cos\,\beta \end{bmatrix}$

$= \begin{bmatrix} cos\,(\alpha+\beta) & -sin\,(\alpha+\beta) \\ sin\,(\alpha+\beta) & cos\,(\alpha+\beta) \end{bmatrix}$

$= A(\alpha + \beta)$

If $\alpha + \beta = 2n\pi$, $A(\alpha) A(\beta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ ($cos2n\pi = 1$ and $sin2n\pi = 0$)

$= I_2$

Exercise 4.2

1. If $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, then prove that $A(B + C) = AB + AC$.

2. Find a, b, c, d, if $\begin{bmatrix} a+b & 4 \\ 3 & c+d \end{bmatrix} + \begin{bmatrix} 6 & a \\ 2d & -1 \end{bmatrix} = \begin{bmatrix} 3b & 3a \\ 3d & 3c \end{bmatrix}$.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & -4 \\ -2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, then prove that $A(B - C) = AB - AC$.

4. If $A = [1 \quad -1 \quad 2]$, $B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, obtain AB and BA, if possible.

5. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A$.

6. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, find $A^2 - 5A$.

7. If $A = \begin{bmatrix} 0 & -tan\frac{\theta}{2} \\ tan\frac{\theta}{2} & 0 \end{bmatrix}$, then prove that $(I_2 - A)\begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix} = I_2 + A$.

8. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then obtain A^2.

9. Obtain X and Y if $X + Y = A = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 5 & 9 \\ 3 & 4 & 8 \end{bmatrix}$, where X is a symmetric and Y is a skew-symmetric matrix.

10. Find a 2×2 matrix X such that $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$.

11. Find real numbers x and y such that $(xI + yA)^2 = A$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

12. Find x if $[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$

13. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove by the principle of mathematical induction that

$A^n = \begin{bmatrix} 2n+1 & -4n \\ n & 1-2n \end{bmatrix}$, $n \in N$.

*

4.4 The Determinant of a Square Matrix :

If all the entries of a square matrix are kept in their respective places and the determinant of this array is taken, then the determinant so obtained is called the determinant of the given square matrix. If A is a square matrix, then determinant of A is denoted by | A | or *det*A.

For instant, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant is $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

If $A = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = 2 - 15 = -13$.

Theorem 4.1 : For square matrices A and B, | AB | = | A | | B |.

We will accept this theorem without proof.

Example 17 : Find | A |, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & sin\,\theta & -cos\,\theta \\ 0 & cos\,\theta & sin\,\theta \end{bmatrix}$.

Solution : $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & sin\,\theta & -cos\,\theta \\ 0 & cos\,\theta & sin\,\theta \end{vmatrix} = sin^2\theta + cos^2\theta = 1$.

Adjoint of a Matrix : For a given square matrix A, if we replace every entry in A by its cofactor as in | A | and then the transpose of this matrix is taken, then the matrix so obtained is called the adjoint of A and is denoted by *adj*A.

If $A = [a_{ij}]_{n \times n}$, then $adjA = [A_{ji}]_{n \times n}$ where A_{ji} is the cofactor of the element a_{ji}.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$.

Example 18 : For $A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$, find *adj*A.

Solution : We take $A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

then $A_{11} = a_{22} = 5$, $A_{12} = -a_{21} = -1$, $A_{21} = -a_{12} = -2$ and $A_{22} = a_{11} = 4$

$\therefore \quad adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$.

[**Note :** To obtain the adjoint of 2 × 2 matrix, interchange the elements on the principal diagonal and change the sign of the elements on the secondary diagonal. e.g. if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.]

Example 19 : Find *adj*A for $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

We have $A_{11} = -1$ $\quad A_{12} = -8$ $\quad A_{13} = -10$

$A_{21} = -5$ $\quad A_{22} = -6$ $\quad A_{23} = 1$

$A_{31} = -1$ $\quad A_{32} = 9$ $\quad A_{33} = 7$

$\therefore \quad adjA = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$.

4.5 Inverse of a Matrix :

For an $n \times n$ square matrix A, if there exists another $n \times n$ square matrix B, such that $AB = I_n = BA$ (I is an identity matrix), then B is called an inverse matrix of A. Inverse of A is denoted by A^{-1}.

It is clear that if B is an inverse of A, then A is an inverse of B.

Theorem 4.2 : If inverse of matrix A exists, then it is unique.

Proof : If possible suppose B and C both are inverses of A.

$\therefore \quad AB = I = BA$ and $AC = I = CA$.

Now $AB = I$

$\therefore$ C(AB) = CI

$\therefore$ (CA)B = C

$\therefore$ IB = C

$\therefore$ B = C

This shows that A has a unique inverse matrix.

> **Note :** Remember in chapter 1, we had seen that for an associative binary operation with identity, inverse is unique. Matrix multiplication of $n \times n$ matrices is associative and has identity I_n.

Theorem 4.3 : For a square matrix A, A(*adj*A) = (*adj*A)A = | A | I.

Proof : We will prove this result for a 3 × 3 square matrix A.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$.

$$\text{Now, } A(adjA) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} \qquad \text{(by the theorems on determinant)}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= |A| I_3$$

Similarly, we can prove that $(adjA)A = |A| I_3$.

Non-singular Matrix : A square matrix is said to be non-singular, if it has an inverse matrix.

[**Note :** If A is a non-singular matrix, then A^{-1} is also non-singular matrix and $(A^{-1})^{-1} = A$.]

Singular Matrix : A matrix which is not non-singular is called a singular matrix.

Theorem 4.4 : A square matrix A is non-singular if and only if $|A| \neq 0$.

Proof : Suppose A is a non-singular matrix and let B be the inverse of A.

$\therefore$ AB = I

$\therefore$ $|AB| = |I|$

$\therefore$ $|A||B| = 1 \neq 0$

$\therefore$ $|A| \neq 0$

Conversely, let $|A| \neq 0$. So $\frac{1}{|A|}$ exists.

Let $B = \frac{1}{|A|} \, adj A$

Then $AB = A\left(\frac{1}{|A|} adj\, A\right) = \frac{1}{|A|}(A\,adj A) = \frac{1}{|A|}\,|A|\,I.$

$\therefore$ AB = I

Similarly, we can prove that BA = I.

$\therefore$ B is the inverse of A.

$\therefore$ A is a non-singular matrix.

(Note : Inverse of matrix A is $A^{-1} = \frac{1}{|A|} adj A$, if it exists.)

Example 20 : Find the inverse of $A = \begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}$, if it exists.

Solution : Here $|A| = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 8 + 15 = 23 \neq 0.$

$\therefore$ A^{-1} exists.

Now, $adj A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$

So, $A^{-1} = \frac{1}{|A|} adj A$

$= \frac{1}{23}\begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} \frac{4}{23} & \frac{3}{23} \\ \frac{-5}{23} & \frac{2}{23} \end{bmatrix}$

Example 21 : Find A^{-1}, if $A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$.

Solution : $|A| = \begin{vmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{vmatrix} = 5(-2 - 3) - 8(0 - 4) + 1(0 - 8)$

$= -25 + 32 - 8$

$= -1 \neq 0$

$\therefore$ A^{-1} exists.

$adj A = \begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|} adj A$

$= \frac{1}{-1}\begin{bmatrix} -5 & 11 & 6 \\ 4 & -9 & -5 \\ -8 & 17 & 10 \end{bmatrix}$

$= \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}$

Some Important Results :

(1) For a square non-singular matrix A, the value of the reciprocal of the determinant of A is the same as the value of the determinant of the inverse of A.

This means $|A^{-1}| = |A|^{-1}$.

Proof : A is a non-singular matrix. Hence $|A| \neq 0$ and A^{-1} exists.

So, $AA^{-1} = I$

So, $|AA^{-1}| = |I|$

$\therefore$ $|A||A^{-1}| = 1$

$\therefore$ $|A^{-1}| = \frac{1}{|A|}$ $\qquad (|A| \neq 0)$

$\therefore$ $|A^{-1}| = |A|^{-1}$

(2) If A and B are non-singular matrices, then AB is also non-singular and

$(AB)^{-1} = B^{-1}A^{-1}$.

Proof : A and B are non-singular, so A^{-1} and B^{-1} exist and $|A| \neq 0$, $|B| \neq 0$.

$\therefore$ $|A||B| \neq 0$

$\therefore$ $|AB| \neq 0$

$\therefore$ AB is a non-singular matrix.

Again, $(AB)(B^{-1}A^{-1}) = A(B(B^{-1}A^{-1}))$

$= A((BB^{-1})A^{-1})$

$= A(IA^{-1})$

$= AA^{-1}$

$= I$

Similarly, we can prove $(B^{-1}A^{-1})(AB) = I$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$

(3) For $m \times n$ matrices A and B, $(AB)^T = B^TA^T$.

We shall accept this result without giving proof.

(4) A^T is non-singular if and only if A is non-singular and $(A^T)^{-1} = (A^{-1})^T$.

Proof : A is a non-singular matrix $\Leftrightarrow |A| \neq 0$

$\Leftrightarrow |A^T| \neq 0$ $\qquad (|A| = |A^T|)$

$\Leftrightarrow A^T$ is non-singular.

Again, $AA^{-1} = A^{-1}A = I$

So, $(AA^{-1})^T = (A^{-1}A)^T = I^T$

$\therefore$ $(A^{-1})^T A^T = A^T (A^{-1})^T = I$ $\qquad (I^T = I)$

$\therefore$ $(A^T)^{-1} = (A^{-1})^T$

(5) $adj A^T = (adj A)^T$

Proof : Let $A = [a_{ij}]$

$\therefore \quad A^T = [a_{ji}]$

$\therefore \quad adj A^T = [A_{ij}]$ (i)

But $adj A = [A_{ji}]$

$\therefore \quad (adj A)^T = [A_{ij}]$ (ii)

From (i) and (ii), we get $adj A^T = (adj A)^T$

4.6 Row Reduced Echelon Form

We have seen some operations like R_{ij}, $R_i(k)$ and $R_{ij}(k)$ as applied to a determinant. Similar operations for columns also can be applied.

The application being similar, we will consider row operations.

(1) If the operation R_{ij} is applied to identity matrix I_n, the resulting matrix is called an elementary matrix E_{ij}.

(2) If the operation $R_i(k)$ is applied to identity matrix I_n, the resulting matrix is called an elementary matrix $E_i(k)$.

(3) If the operation $R_{ij}(k)$ is applied to identity matrix I_n, the resulting matrix is called an elementary matrix $E_{ij}(k)$.

Applying R_{12} to matrix A is the same as finding product E_{12} A for any matrix A.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 2 & 1 & 6 \end{bmatrix}$$

$$R_{12} \text{ gives } \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \\ 2 & 1 & 6 \end{bmatrix} \qquad \text{(i)}$$

$$\text{Also } E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{12} A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 3 \\ 2 & 1 & 6 \end{bmatrix} \qquad \text{(ii)}$$

(i) and (ii) prove our assertion.

Similarly any elementary operation R_{ij}, $R_i(k)$ or $R_{ij}(k)$ on matrix A is equivalent to premultiplying A by E_{ij}, $E_i(k)$ or $E_{ij}(k)$ respectively.

For column operations post-multiplication has to be carried out.

Now we define a reduced row echelon matrix. A matrix is in reduced row echelon form if

(1) **The first non-zero entry of each row called the leading entry is 1.**

(2) **Each leading entry is in a column to the right of the leading entry of the previous row.**

(3) **A row with all entries zero is called a zero row. All zero rows occur below rows with at least one entry non-zero (called a non-zero row).**

(4) The leading entry is the only non-zero entry in its column

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **are in reduced row echelon form.**

A result : The row reduced form of a non singular matrix is I_n.

We can obtain inverse of a non-singular matrix as follows :

Write A = IA.

Apply elementary row operations on A and I so that A on left-hand side is converted to its reduced row echelon form namely I_n (being non-singular).

Then, we will have an equation like this I = PA.

where I gets converted to P by elementary row operations same as on left-hand side matrix A.

Then $P = A^{-1}$.

How to get row reduced echelon form of a matrix A ?

(1) (a) Find the pivot, the first non-zero entry in the first column.

For, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 2 \\ 3 & 4 & 5 \end{bmatrix}$. 1 is the pivot.

(b) If necessary interchange rows so that the leading entry in the first row is non-zero.

$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$. To have pivot in the first row, we will apply R_{12} or R_{13}.

For instant, if we apply R_{13}, then in $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$, we will get 1 3 3 as a first row with 1 as a pivot.

(c) Multiply each element in the pivot row by inverse (reciprocal) of the leading entry, so that leading entry becomes 1.

In $\begin{bmatrix} 3 & 5 & 1 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ leading entry is 3. So we multiply each element of the first row by $\frac{1}{3}$ to get 1 $\frac{5}{3}$ $\frac{1}{3}$ as the first row.

So the matrix will be $\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$.

(d) Add multiples of the pivot row to each of lower rows so that every element in the pivot column of lower rows becomes 0.

We apply $R_{12}(-2)$, $R_{13}(-4)$ to the matrix which we have at the end of (c). The matrix will become $\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} \\ 0 & -\frac{7}{3} & \frac{7}{3} \\ 0 & \frac{-17}{3} & \frac{2}{3} \end{bmatrix}$ with first column $\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$.

(2) (a) Repeat the above procedure from step (1) ignoring previous pivot row.

(b) Continue till there are no more leading entries to be processed.

(c) Now the matrix becomes a triangular matrix having zeroes below principal diagonal. After performing some operations on the matrix obtained in (1)(d), we have matrix as $\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(3) (a) Identify the last row having leading entry equal to 1. Call it the pivot row now.

(b) Add multiples of this pivot row to each of the upper rows untill every element above the pivot becomes 0.

(c) Moving up the matrix repeat this process for each row.

Now performing $R_{31}\left(-\frac{1}{3}\right)$ and $R_{32}(1)$ we have,

$$\begin{bmatrix} 1 & \frac{5}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now applying $R_{21}\left(-\frac{5}{3}\right)$, we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ i.e., we get I_3.

Thus performing operations on $A = IA$, we get $I = PA$. Here $P = A^{-1}$.

Let us understand by an example.

Example 22 : Find inverse of $\begin{bmatrix} 0 & -1 & 1 \\ 3 & -3 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ by elementary row operations.

Solution : $\begin{bmatrix} 0 & -1 & 1 \\ 3 & -3 & 4 \\ 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 3 & -3 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 & -3 & 4 \\ 3 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$ **(R_{31}) (To bring leading entry non-zero)**

$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 3 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$ **$R_1\left(\frac{1}{2}\right)$ (To make leading entry 1)**

$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & \frac{3}{2} & -2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \\ 1 & 0 & 0 \end{bmatrix} A$ **$R_{12}(-3)$**

$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & -1 \\ 1 & 0 & 0 \end{bmatrix} A$ **$R_2\left(\frac{2}{3}\right)$ (Leading element of second row is made 1)**

$$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & -1 \\ 1 & \frac{2}{3} & -1 \end{bmatrix} A \qquad R_{23}(1)$$

$$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 2 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & -1 \\ -3 & -2 & 3 \end{bmatrix} A \qquad R_3(-3) \text{ (Leading element of third row is made 1)}$$

$$\therefore \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -\frac{11}{2} \\ -4 & -2 & 3 \\ -3 & -2 & 3 \end{bmatrix} A \qquad R_{32}\left(\frac{4}{3}\right), R_{31}(-2)$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -4 & -2 & 3 \\ -3 & -2 & 3 \end{bmatrix} A \qquad R_{21}\left(\frac{3}{2}\right)$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -4 & -2 & 3 \\ -3 & -2 & 3 \end{bmatrix}$$

Example 23 : By using elementary operations, find the inverse of $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

Solution : We take A = IA.

We shall use elementary row operations on this matrix equation.

$$\therefore \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\therefore \begin{bmatrix} 1 & 4 \\ 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A \qquad R_{12}(-3)$$

$$\therefore \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{10} & \frac{-1}{10} \end{bmatrix} A \qquad R_2\left(-\frac{1}{10}\right)$$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{10} & \frac{4}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix} A \qquad R_{21}(-4)$$

Thus, $A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$

Example 24 : Obtain the inverse of matrix $A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$ by reduced row echelon method.

Solution : We write, A = IA

$$\begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad (R_{31}(-1))$$

$$\therefore \begin{bmatrix} 1 & 5 & 2 \\ 0 & 2 & 1 \\ 0 & -17 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix} A \qquad (R_{13}(-4))$$

$$\therefore \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 1 \\ 0 & -17 & -9 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{2} & -1 \\ 0 & 1 & 0 \\ -4 & 0 & 5 \end{bmatrix} A \qquad \left(R_{21}\left(-\frac{5}{2}\right)\right)$$

$$\therefore \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & -17 & -9 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{2} & -1 \\ 0 & \frac{1}{2} & 0 \\ -4 & 0 & 5 \end{bmatrix} A \qquad \left(R_{2}\left(\frac{1}{2}\right)\right)$$

$$\therefore \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{2} & -1 \\ 0 & \frac{1}{2} & 0 \\ -4 & \frac{17}{2} & 5 \end{bmatrix} A \qquad (R_{23}(17))$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 & -11 & -6 \\ 0 & \frac{1}{2} & 0 \\ -4 & \frac{17}{2} & 5 \end{bmatrix} A \qquad (R_{31}(-1))$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ -4 & \frac{17}{2} & 5 \end{bmatrix} A \qquad (R_{32}(1))$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix} A \qquad (R_{3}(-2))$$

$$\therefore I = A^{-1}A, \text{ where } A^{-1} = \begin{bmatrix} 5 & -11 & -6 \\ -4 & 9 & 5 \\ 8 & -17 & -10 \end{bmatrix}.$$

Example 25 : Find inverse of $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix}$ by reduced row echelon method.

Solution : We write $\begin{bmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

$$\therefore \quad \begin{bmatrix} 1 & 5 & 2 \\ 0 & -4 & 5 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_{12}(-1)$$

$$\therefore \quad \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -\frac{5}{4} \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2\left(-\frac{1}{4}\right)$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & \frac{33}{4} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{5}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{3}{4} & -\frac{3}{4} & 1 \end{bmatrix} \qquad R_{21}(-5),\ R_{23}(3)$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & \frac{33}{4} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{5}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 3 & -3 & 4 \end{bmatrix} \qquad R_3(4)$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix} A \qquad R_{32}\left(\frac{5}{4}\right),\ R_{31}\left(\frac{-33}{4}\right)$$

$\therefore \quad I_3 = PA$

$$\therefore \quad A^{-1} = \begin{bmatrix} -25 & 26 & -33 \\ 4 & -4 & 5 \\ 3 & -3 & 4 \end{bmatrix}$$

Unique Solution of a System of Linear Equations Using Inverse of a Matrix :

Suppose, $a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

is a system of three linear equations in x, y, z.

If we take, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

then the system of equations can be written as, AX = B.

If A is a non-singular matrix, then A^{-1} exists.

Now, AX = B

$\therefore \quad A^{-1}(AX) = A^{-1}B$

$\therefore \quad (A^{-1}A)X = A^{-1}B$

$\therefore \quad IX = A^{-1}B$

$\therefore \quad X = A^{-1}B$

Suppose, $A^{-1}B = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$.

Thus, $x = p_1$, $y = p_2$, $z = p_3$, is the unique solution of the given system of linear equations.

[**Note :** This result is also true for a system of two linear equations in two unknowns.]

Example 26 : Using matrix method, solve : $x - 2y = 4$ and $-3x + 5y = -7$.

Solution : The system can be expressed as $\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

or AX = B, where $A = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0$

$\therefore$ A^{-1} exists.

Hence, the system has a unique solution given by $A^{-1}B = X$.

Now, $adj A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

So, $A^{-1} = \frac{1}{|A|} adj A$

$= \frac{1}{-1}\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$

Now, $X = A^{-1}B$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -20+14 \\ -12+7 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$

$\therefore$ $x = -6$, $y = -5$ is the required solution.

Example 27 : If the system of equations $x + y + z = 3$, $2x - y - z = 3$, $x - y + z = 9$ has unique solution, then find it.

Solution : The system of equations can be expressed in the matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$, then the system of equations is AX = B.

Now $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1(-1-1) - 1(2+1) + 1(-2+1)$

$= -2 - 3 - 1 = -6 \neq 0$

$\therefore$ A^{-1} exists and hence the given system has a unique solution.

Now, $adj A = \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|} adj A$

$= \frac{1}{-6} \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix}$

Now, $X = A^{-1}B$

$\therefore$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-6} \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$

$= \frac{1}{-6} \begin{bmatrix} -6+(-6)+0 \\ -9+0+27 \\ -3+6-27 \end{bmatrix}$

$= \frac{1}{-6} \begin{bmatrix} -12 \\ 18 \\ -24 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$

So, $x = 2$, $y = -3$ and $z = 4$.

Exercise 4.3

1. Find the adjoint for the following matrices :

(1) $\begin{bmatrix} 5 & -2 \\ 1 & -3 \end{bmatrix}$ (2) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ (3) $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1} if it exists.

3. If $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$, prove that $A^{-1} = A^T$.

4. If $A = \begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, verify $(AB)^{-1} = B^{-1}A^{-1}$.

5. For $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, show that $adj(adjA) = A$.

6. If $A = \begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 9 \\ 7 & 8 \end{bmatrix}$, then verify $(AB)^{-1} = B^{-1}A^{-1}$.

7. Find $x \in R$ if $A = \begin{bmatrix} 5x & 10 \\ 8 & 7 \end{bmatrix}$ and $|A| = 25$.

8. **By using reduced row echelon method, find the inverse of the following matrices :**

(1) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ (3) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

9. **Solve the system of equations by matrix method :**

(1) $3x + 4y + 5 = 0$
$11x - 2y = 15$

(2) $5x - 7y = 2$
$7x - 5y = 3$

10. **Use matrix method to solve the following system of equations :**

(1) $4x - 3y + 2z = 4$
$3x - 2y + 3z = 8$
$4x + 2y - 2z = 2$

(2) $x + 2y + z = 4$
$x - y - z = 0$
$-x + 3y - z = -2$

*

Miscellaneous Examples :

Example 28 : For $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, prove that $A^2 - 4A + 7I_2 = O$ and hence obtain A^{-1}.

Solution : Now, $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad A^2 - 4A + 7I_2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= O$$

Here, $|A| = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$

$\therefore$ A is a non-singular matrix. Hence A^{-1} exists.

Now, multiplying $A^2 - 4A + 7I_2 = O$ by A^{-1} on both the sides, we get,

$$A^{-1}(A^2 - 4A + 7I_2) = A^{-1}O$$

$\therefore$ $A^{-1}(A^2) - 4(A^{-1}A) + 7(A^{-1}I_2) = O$

$\therefore$ $(A^{-1}A)A - 4I + 7A^{-1} = O$

$\therefore$ $IA - 4I + 7A^{-1} = O$

$\therefore$ $7A^{-1} = 4I - A$

$\therefore$ $A^{-1} = \frac{1}{7}(4I - A)$

$$= \frac{1}{7}\left\{4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{7}\left\{\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -2 & -3 \\ 1 & -2 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

Example 29 : If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then prove that $A^2 - 4A - 5I_3 = O$ and hence obtain A^{-1}.

Solution : $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$\therefore$ $A^2 - 4A - 5I_3 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= O$$

Now, $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(-3) - 2(-2) + 2(2)$

$$= -3 + 4 + 4$$

$$= 5 \neq 0$$

$\therefore \quad A^{-1}$ exists.

Now, multiplying $A^2 - 4A - 5I_3 = O$ by A^{-1} on both the sides,

we have,

$\therefore \quad A^{-1}(A^2) - 4(A^{-1}A) - 5(A^{-1}I_3) = A^{-1}O$

$\therefore \quad (A^{-1}A)A - 4I_3 - 5A^{-1} = O$

$\therefore \quad I_3 A - 4I_3 = 5A^{-1}$

$\therefore \quad A - 4I_3 = 5A^{-1}$

$\therefore \quad A^{-1} = \frac{1}{5}(A - 4I_3)$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} 1-4 & 2+0 & 2+0 \\ 2+0 & 1-4 & 2+0 \\ 2+0 & 2+0 & 1-4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

Example 30 : Find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix} = -\cos^2\alpha - \sin^2\alpha = -1 \neq 0$$

$\therefore \quad A^{-1}$ exists.

Cofactors of the elements of A are,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -\cos^2\alpha - \sin^2\alpha = -1$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 0$$

Similarly, $A_{21} = 0$, $A_{22} = -\cos\alpha$, $A_{23} = -\sin\alpha$

$A_{31} = 0$, $A_{32} = -\sin\alpha$, $A_{33} = \cos\alpha$

$$\therefore \quad adj A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} adj A$$

$$= \frac{1}{-1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

[**Note :** If $A^{-1} = A$, then such a matrix said to be an idempotent matrix.]

Example 31 : Find the equations of lines passing through (2, −1) (4, 0) and (−1, −2), (4, 1) using determinant method. Find the point of intersection (if it exists) using matrix method.

Solution : The equation of the line passing through (2, −1) and (4, 0) is $\begin{vmatrix} x & y & 1 \\ 2 & -1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 0.$

$\therefore \quad x(-1) - y(-2) + 4 = 0$

$\therefore \quad -x + 2y + 4 = 0$

$\therefore \quad x - 2y = 4$

The equation of the line passing through (−1, −2) and (4, 1) is $\begin{vmatrix} x & y & 1 \\ -1 & -2 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 0$

$\therefore \quad x(-3) - y(-5) + 7 = 0$

$\therefore \quad -3x + 5y = -7$

$\therefore \quad 3x - 5y = 7$

$\therefore$ The equations of lines are $x - 2y = 4$

$3x - 5y = 7$

The system of equations can be written in the matrix form as,

$$\begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

or AX = B, where $A = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 1 & -2 \\ 3 & -5 \end{vmatrix} = -5 + 6 = 1 \neq 0$

∴ A^{-1} exists.

$$adj A = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}. \text{ Hence } A^{-1} = \frac{1}{|A|} adj A = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -20 + 14 \\ -12 + 7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ -5 \end{bmatrix}$$

∴ $x = -6$ and $y = -5$.

∴ The point of intersection of the two lines is (–6, –5).

Example 32 : Does the system of simultaneous linear equations,

$x + 3y + 4z = 8$, $2x + y + 2z = 5$, $5x + y + z = 7$ have unique solution ?

If so, find it using matrix method.

Solution : Writing $x + 3y + 4z = 8$

$2x + y + 2z = 5$

$5x + y + z = 7$ in the matrix form as

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

The system is AX = B.

$$\text{Now, } |A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix} = 1(-1) - 3(-8) + 4(-3)$$

$$= -1 + 24 - 12$$

$$= 11 \neq 0$$

∴ A^{-1} exists.

∴ The system has a unique solution.

Now, taking the matrix $A = [a_{ij}]_{3 \times 3}$, we have cofactors of the entries of A as,

$A_{11} = -1,\ A_{12} = 8,\ A_{13} = -3$

$A_{21} = 1,\ A_{22} = -19,\ A_{23} = 14$

$A_{31} = 2,\ A_{32} = 6,\ A_{33} = -5$

$$\therefore \quad adj A = \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|} adj A$

$$\therefore \quad A^{-1} = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

As, $X = A^{-1}B$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -8+5+14 \\ 64-95+42 \\ -24+70-35 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 1,\ y = 1,\ z = 1.$

Exercise 4

1. If $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$, prove that $A^{-1} = A^T$. Also find AA^T.

2. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, prove that $A^{-1} = \frac{1}{19}A$

3. If $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$, find $(AB)^{-1}$.

4. If $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 2 & 1 \\ 4 & 3 & -1 \end{bmatrix}$, find $(AB)^{-1}$.

5. If $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, find $B^{-1}AB$.

6. Prove that If $A^2 - 6A + 17I_2 = O$, where $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ and hence find A^{-1}.

7. If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, prove that $A^{-1} = A^2$.

8. For $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, prove that If $A^3 - 6A^2 + 5A + 11I_3 = O$. Using this matrix relation, obtain A^{-1}.

9. If $A = \begin{bmatrix} 3 & 0 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ -4 & 3 \end{bmatrix}$, then obtain $A^2 + AB + 6B$ without multiplying the given matrices.

10. Solve the system of equations by matrix method (if unique solution exists).

(1) $3x - 5y = 1, x + 2y = 4$ (2) $3x + 4y - 5 = 0, y - x - 3 = 0$

11. If the following system of equations has unique solution, then find the solution set :

(1) $2x + y + z = 2$

$x + 3y - z = 5$

$3x + y - 2z = 6$

(2) $\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10$

$\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13$ $(xyz \neq 0)$

12. For $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$, find $(a^2 + bc + 1)I_2 - aA^{-1}$.

13. Two intersecting lines have slopes m_1 and m_2 and their y-intercepts are c_1 and c_2 $(m_1 \neq m_2)$ respectively. Using matrix, find their point of intersection.

14. Find $x \in R$, if $A = \begin{bmatrix} 2x & 9 \\ -3 & -2 \end{bmatrix}$ and $|A| = 3$.

15. Find $x \in R$, if $[x \quad -5 \quad -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$.

16. Express $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 4 & -1 & 5 \end{bmatrix}$ as a sum of a symmetric matrix and a skew-symmetric matrix.

17. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, prove $AA^T = I$. Deduce $A^{-1} = A^T$.

18. If for square matrices A and B, AB = A and BA = B, prove $A^2 = A$ and $B^2 = B$.

19. If B is a square matrix and $B^2 = B$, then prove that $A = I - B$ satisfies $A^2 = A$ and $AB = BA = O$.

20. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, prove $A^3 = O$. (See that $A^3 = O$, even though $A \neq O$)

21. A is a 3×3 square matrix, prove that, $|adjA| = |A|^2$.

22. Find matrix A and B such that $A \neq O$, $B \neq O$ but $AB = O$.

23. If $A(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, prove $A(\alpha)\,A(-\alpha) = I$.

24. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A (1 Mark)

(1) A is a 3×3 matrix, then $|3A| = ...\ |A|$ ☐

(a) 3 (b) 6 (c) 9 (d) 27

(2) If $A = [a_{ij}]_{n \times n}$ such that $a_{ij} = 0$ for $i \neq j$ then A is ...... $(a_{ii} \neq a_{jj})$ $(n > 1)$ ☐

(a) a column matrix (b) a row matrix (c) a diagonal matrix (d) a scalar matrix

(3) $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, the correct statement is .... ☐

(a) A^{-1} does not exist (b) $A = (-1)\,I_3$

(c) $A^2 = I$ (d) A is a diagonal matrix

(4) A is 3×4 matrix, if A^TB and BA^T are defined then, B is a ...... matrix. ☐

(a) 4×3 (b) 3×3 (c) 4×4 (d) 3×4

(5) If A is skew-symmetric 3×3 matrix, $|A| = ...$. ☐

(a) 1 (b) 0 (c) −1 (d) 3

Section B (2 Marks)

(6) The system of equations $ax + y + z = a - 1$, $x + ay + z = a - 1$ and $x + y + az = a - 1$ does not have unique solution if a = ...

(a) 1 or −2 (b) 3 (c) 2 (d) −1

(7) If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$, then x = ..., y = ...

(a) $x = a^2 + b^2$, $y = a^2 - b^2$ (b) $x = 2ab$, $y = a^2 + b^2$

(c) $x = a^2 + b^2$, $y = ab$ (d) $x = a^2 + b^2$, $y = 2ab$

(8) If α and β are not the multiple of $\frac{\pi}{2}$ and

$\begin{bmatrix} \cos^2\alpha & \cos\alpha \sin\alpha \\ \cos\alpha \sin\alpha & \sin^2\alpha \end{bmatrix} \times \begin{bmatrix} \cos^2\beta & \sin\beta \cos\beta \\ \sin\beta \cos\beta & \sin^2\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\alpha - \beta$ is ....

(a) any multiple of π (b) odd multiple of $\frac{\pi}{2}$

(c) 0 (d) odd multiple of π

(9) If $\begin{bmatrix} x & 0 \\ 1 & y \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$, then x = ..., y = ...

(a) $x = 3$, $y = 2$ (b) $x = 3$, $y = -2$ (c) $x = -3$, $y = -2$ (d) $x = -3$, $y = 2$

(10) If inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ is $\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$, then α = ...

(a) 5 (b) −5 (c) 2 (d) −2

Section C (3 Marks)

(11) If AB = BA and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then B = ...

(a) $\begin{bmatrix} x & x \\ y & 0 \end{bmatrix}$ (b) $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ (c) $\begin{bmatrix} x & y \\ 0 & y \end{bmatrix}$ (d) $\begin{bmatrix} x & x \\ 1 & x \end{bmatrix}$

(12) If $A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ and $A^2 - kA - 5I = O$, then k = ...

(a) 3 (b) 7 (c) 5 (d) 9

(13) If $[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = O$, then x = ...

(a) $\frac{-9 \pm \sqrt{35}}{2}$ (b) $\frac{-7 \pm \sqrt{53}}{2}$ (c) $\frac{-9 \pm \sqrt{53}}{2}$ (d) $\frac{-7 \pm \sqrt{35}}{2}$

(14) Matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ if $AA^T = I$, then $(x, y, z) = (....., ..., ...)$. $(x, y, z > 0)$ ☐

(a) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right)$ (b) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right)$ (d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$

Section D (4 Marks)

(15) If $A \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, then A = ... ☐

(a) $\begin{bmatrix} 2 & -1 & 1 \\ 0 & -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -2 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$

(16) If $A = \begin{bmatrix} cos\frac{2\pi}{3} & -sin\frac{2\pi}{3} \\ sin\frac{2\pi}{3} & cos\frac{2\pi}{3} \end{bmatrix}$, then A^3 = ... ☐

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

(17) Check, whether $\frac{1}{11}\begin{bmatrix} -1 & 8 & \alpha \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, if so, then α = ... ☐

(a) −3 (b) 2 (c) −5 (d) not exists.

*

Summary

We have studied the following points in this chapter :

1. **Matrix** : Any rectangular arrangement or an array of numbers enclosed in brackets such as [ ] or ( ) is called a matrix. The numbers are the elements of the matrix.
2. If two matrices have same order and corresponding elements are same in both the matrices, then they are equal matrices. $A = B \Rightarrow [a_{ij}] = [b_{ij}] \Leftrightarrow a_{ij} = b_{ij} \; \forall i, j$
3. **Types of matrices** : Row matrix, Column matrix, Square matrix, Diagonal matrix, Zero matrix.
4. **Sum of two matrices** : Two matrices must have the same number of rows and the same number of columns, otherwise it is not possible to add the matrices.

   $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

5. **Properties of Matrix Addition :**

(1) Commutative Law for Addition

(2) Associative Law for Addition

(3) The Identity for Addition of Matrices

(4) Existence of Additive Inverse

6. **Product of a Matrix with a Scalar and Properties :**

(1) If A = $[a_{ij}]_{m \times n}$ and $k \in$ R, then for $k \in$ R, kA = $[ka_{ij}]_{m \times n}$.

(2) k(A + B) = kA + kB where A, B are matrices and k, $l \in$ R

(3) (kl)A = $k(l$A)

(4) 1A = A

(5) (−1)A = −A

7. **Transpose of a Matrix :** A = $[a_{ij}]_{m \times n}$, then transpose of A is A^T = A' = $[a_{ji}]_{n \times m}$.

8. **Symmetric Matrix :** For a square matrix A, if A^T = A, then A is called a symmetric matrix.

9. **Skew-Symmetric Matrix :** For a square matrix A, if $A^T = -A$, then A is called a skew-symmetric matrix.

10. (1) $(A + B)^T = A^T + B^T$, (2) $(A^T)^T = A$, (3) $(kA)^T = kA^T$

11. **Multiplication of two matrics :** If the number of columns of A = the number of rows of B, then the product AB is possible.

12. **Identity (unit) matrix :** In a square matrix, if all elements on principal diagonal are 1 and the rest are 0, then the matrix is called an identity matrix, denoted by I.

13. Determinant of a square matrix A is deonted by | A |.

14. | AB | = | A | | B | where A and B are square matrices.

15. **Adjoint of a matrix :** If we replace every entry of a square matrix A by its cofactor and then tranpose of this is taken, then the matrix so obtained is the adjoint of A denoted by *adj*A.

16. **Inverse of a matrix :** For two square matrices A and B; if AB = BA = I, then they are inverse of each other.

17. **Non-singular matrix :** If the inverse matrix of a square matrix exists, then that matrix is called a non-singular matrix. Determinant of a non-singular matrix is a non-zero real number.

18. Inverse of A is $A^{-1} = \frac{1}{|A|}(adjA)$; $|A| \neq 0$

19. A^{-1} can be obtained by elementary rows (or column) operations on the matrix A. (Symbols of the operations are as determinant.)

20. **Echelon Method of finding inverse of a matrix :** Take matrix equation A = IA, now apply a sequence of elementary row (or column) operations on A on L.H.S. and same to I, then A of L.H.S. will be converted into I and I on R.H.S. will become A^{-1} as $I = A^{-1}A$. This method of finding inverse of matrix is called reduced row echelon method.

21. Solution of a system of simultaneous linear equations can be obtained by matrix.