Continuity and Differentiability

5

Do not worry about your difficulties in mathematics.
I assure you that mine are greater.

– Albert Einstein

The last thing one knows when writing a book is what to put first.

– Blaise Pascal

5.1 Introduction

We introduced the idea of limit in standard XI. An intuitive approach and graphical understanding helped us to grasp the idea of limit. At several places, we mentioned the word 'continuous'. What is a 'continuous function' ? We will now try to learn the concept of continuity which is very useful to study limits and it links limits and differentiability. Look at the graph of $f(x) = [x]$, $x \in R$.

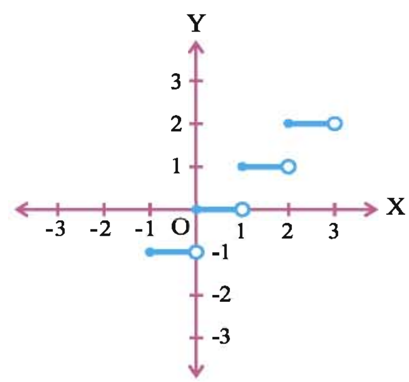

Figure 5.1

We cannot draw the graph of the function without lifting the pencil from the plane of the paper. At every point on the graph, with integer x-coordinate, this situation arises. The same is the situation with the graph of signum function

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

or

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

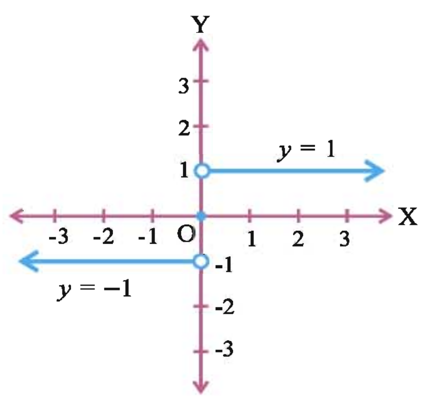

Figure 5.2

At $x = 0$, the graph 'jumps'.

Here $\lim_{x \to 0-} f(x) = -1$ and $\lim_{x \to 0+} f(x) = 1$.

So, $\lim_{x \to 0} f(x)$ does not exist. In the example of $f(x) = [x]$ also, we infer from the graph, $\lim_{x \to 1-} [x] = 0$, $\lim_{x \to 1+} [x] = 1$.

$\therefore \lim_{x \to 1} [x]$ does not exist.

5.2 Continuity

Consider the function $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 5 & x = 2 \end{cases}$

Hence, $f(x) = \begin{cases} x + 2 & x \neq 2 \\ 5 & x = 2 \end{cases}$

Here, the graph of the function consists of $(\overleftrightarrow{AB} - \{P\}) \cup \{Q\}$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x) = 4$$

$\therefore \quad \lim_{x \to 2} f(x) = 4$

But $f(2) = 5$

$\therefore \quad \lim_{x \to 2} f(x) \neq f(2)$

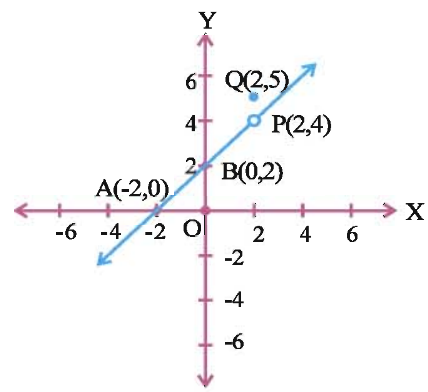

Figure 5.3

Here also the graph of $f(x)$ cannot be drawn without lifting the pencil from the plane of the paper. This is the idea of continuity. The graph 'breaks' or is 'not continuous'.

Let us now give a formal definition.

Continuity : Let f be a function defined on an interval (a, b) containing c. $c \in R$. If $\lim_{x \to c} f(x)$ exists and is equal to $f(c)$, then we say f is continuous at $x = c$.

In other words, if $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and are equal to $f(c)$, we say f is continuous at $x = c$.

$\therefore$ f is continuous at $x = c \Leftrightarrow \lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = f(c)$.

If f is not continuous at $x = c$, we say f is discontinuous at $x = c$.

That f is discontinuous at $x = c$ in a domain may occur in one of the following situations.

(1) $\lim_{x \to c+} f(x)$ or $\lim_{x \to c-} f(x)$ does not exist.

(2) $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist but are unequal.

(3) $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist and are equal.

i.e. $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = \lim_{x \to c} f(x)$

but f is not defined for $x = c$ or $\lim_{x \to c} f(x) \neq f(c)$

If f is defined at an isolated point, we say it is continuous at that point. Consequently a function defined on a finite set $\{x_1, x_2, x_3, ..., x_n\}$ is continuous.

We say f is continuous in a domain, if it is continuous at all points of the domain.

If f is defined on $[a, b]$, then f is continuous on $[a, b]$ if

(1) f is continuous at every point of (a, b)

(2) $\lim_{x \to a+} f(x) = f(a)$ **(f is not defined for $x < a$)**

(3) $\lim_{x \to b-} f(x) = f(b)$ (f is not defined for $x > b$)

Example 1 : Examine the continuity of $f : R \to R$, $f(x) = 2x - 4$ at $x = 3$.

Solution : $f(x) = 2x - 4$ is a polynomial in x.

$\therefore \lim_{x \to 3} f(x) = \lim_{x \to 3} (2x - 4) = 2 \cdot 3 - 4 = 2$

$f(3) = 2 \cdot 3 - 4 = 2$

$\therefore \lim_{x \to 3} f(x) = f(3)$

$\therefore$ f is continuous at $x = 3$.

The graph is a straight line and it is 'unbroken'.

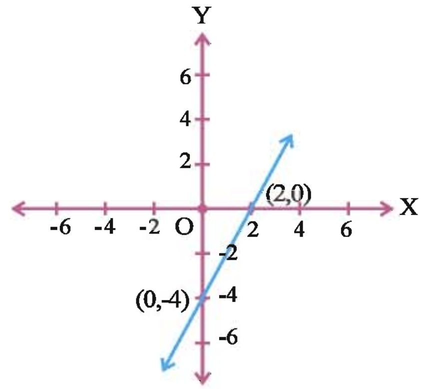

Figure 5.4

Example 2 : Examine continuity of $f : R \to R$, $f(x) = x^2$ at $x = 2$.

Solution : $\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4$, $f(2) = 4$

($f(x) = x^2$ is a polynomial)

$\therefore \lim_{x \to 2} f(x) = f(2)$

$\therefore$ $f(x) = x^2$ is continuous at $x = 2$.

The graph is 'continuous'.

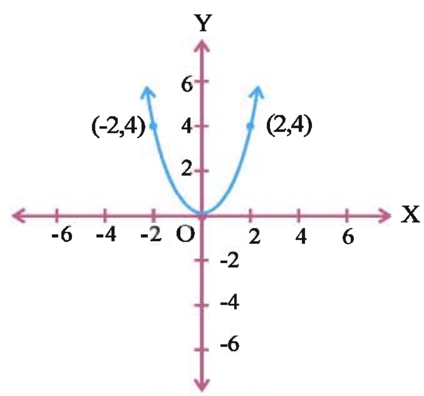

Figure 5.5

Example 3 : Is $f : R \to R$, $f(x) = |x|$ continuous on R?

Solution : Here, we have to examine continuity of $|x|$ on the domain.

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Let $c > 0$. For some $\delta > 0$,

we can have $c - \delta > 0$ (let $\delta = \frac{c}{2}$)

$f(x) = |x| = x$ in $(c - \delta, c + \delta)$ $(c - \delta > 0)$

$\therefore \lim_{x \to c} f(x) = \lim_{x \to c} x = c$, $f(c) = |c| = c$ $(c > 0)$

$\therefore \lim_{x \to c} f(x) = f(c)$

$\therefore$ f is continuous for all $c > 0$

Let $c < 0$. There exists some $\delta > 0$ such that $c + \delta < 0$.

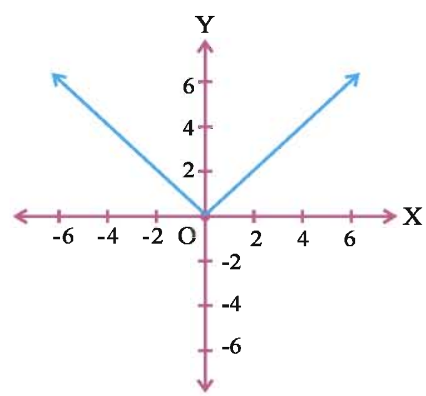

Figure 5.6

$\therefore \quad f(x) = |x| = -x$ in $(c - \delta, c + \delta)$ **$(c + \delta < 0)$**

$\therefore \quad \lim_{x \to c} f(x) = \lim_{x \to c} (-x) = -c, f(c) = |c| = -c$ **$(c < 0)$**

$\therefore \quad \lim_{x \to c} f(x) = f(c)$

$\therefore \quad f$ is continuous for all $c < 0$.

$\therefore \quad \lim_{x \to 0+} f(x) = \lim_{x \to 0+} |x| = \lim_{x \to 0+} x = 0$ **$(x > 0)$**

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} |x| = \lim_{x \to 0-} -x = 0$ **$(x < 0)$**

$f(0) = |0| = 0$

$\therefore \quad \lim_{x \to 0} f(x) = f(0) = 0$

$\therefore \quad f$ is continuous at $x = 0$.

$\therefore \quad f$ is continuous for all $x \in$ R.

Example 4 : Discuss the continuity of constant function $f(x) = k$ on R.

Solution : For $c \in$ R, $\lim_{x \to c} f(x) = \lim_{x \to c} k = k = f(c)$ **$(\lim_{x \to c} k = k)$**

$\therefore$ A constant function is continuous on its domain.

Example 5 : Discuss the continuity at $x = 0$.

$$f(x) = \begin{cases} x^3 + x^2 + x + 1 & x \neq 0 \\ 5 & x = 0 \end{cases}$$

Solution : $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^3 + x^2 + x + 1) = 1$ **(limit of a polynomial)**

$f(0) = 5$

$\therefore \quad \lim_{x \to 0} f(x) \neq f(0)$

$\therefore \quad f$ is discontinuous at $x = 0$

Example 6 : Examine the continuity of the identity function on R.

Solution : Here $f(x) = x$.

Let $a \in$ R.

$\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a)$

$\therefore$ The identity function is continuous on R.

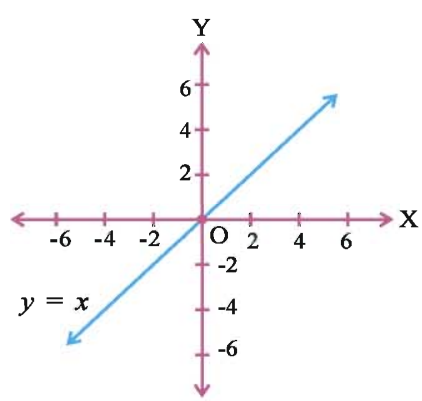

Figure 5.7

Example 7 : Discuss the continuity of $f(x) = \frac{1}{x}$, $x \in R - \{0\}$.

Solution : $f(x) = \frac{1}{x}$ is a rational function.

Let $c \neq 0$.

$$\lim_{x \to c} f(x) = \frac{\lim\limits_{x \to c} 1}{\lim\limits_{x \to c} x} = \frac{1}{c}$$

$$f(c) = \frac{1}{c}$$

$$\lim_{x \to c} f(x) = \frac{1}{c} = f(c)$$

$\therefore$ f is continuous for all $c \in R - \{0\}$.

Note : For $x = 0$, $f(x) = \frac{1}{x}$ is not defined. Let us study behaviour of $f(x)$ near 0.

Let $x > 0$.

x	0.1	0.01	0.001	10^{-n}
$f(x)$	10	$100 = 10^2$	$1000 = 10^3$	10^n

As $x \to 0+$, $f(x)$ increases unboundedly.

In such a case we say $f(x) \to \infty$ as $x \to 0+$. We do not write $\lim\limits_{x \to 0+} f(x) = \infty$.

$\lim\limits_{x \to 0+} f(x)$ does not exit.

Limit of a function is a real number. ∞ is not a real number or it is a member of extended real number system.

Let $x < 0$.

x	-0.1	-0.01	-0.001	-10^{-n}
$f(x)$	-10	$-100 = -10^2$	$-1000 = -10^3$	-10^n

$\therefore$ Here as x decreases $f(x)$ decreases and as $x \to 0-$, $f(x) \to -\infty$.

Again $\lim\limits_{x \to 0-} f(x) = -\infty$ is not to be written.

$\lim\limits_{x \to 0-} f(x)$ **does not exist.**

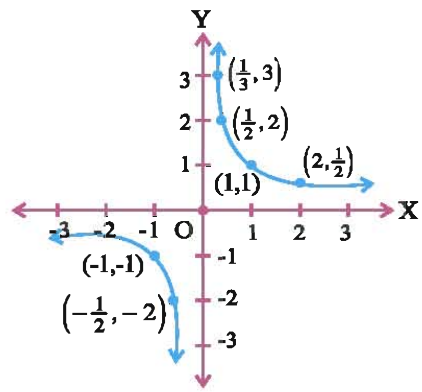

Figure 5.8

Example 8 : $f(x) = \frac{1}{x^2}$, $x \neq 0$. Discuss continuity for $x \in R - \{0\}$.

Solution : Let $c \neq 0$. $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \frac{1}{x^2} = \frac{\lim\limits_{x \to c} 1}{\lim\limits_{x \to c} x^2} = \frac{1}{c^2}$

$\therefore$ f is continuous for $x \in R - \{0\}$

Note : For $x = 0$, $\lim\limits_{x \to 0} \frac{1}{x^2}$ does not exist.

$\frac{1}{x^2} \to \infty$ as $x \to 0$.

x	-0.1	0.1	-0.01	0.01	$\pm 10^{-n}$
$f(x)$	100	100	10000	10000	10^{2n}

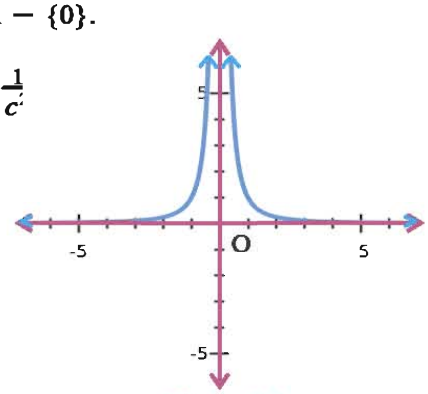

Figure 5.9

Example 9 : Examine the continuity of

$$f(x) = \begin{cases} x + 3 & x < 2 \\ 3 - x & x \geq 2 \end{cases} \quad \text{at } x \in R.$$

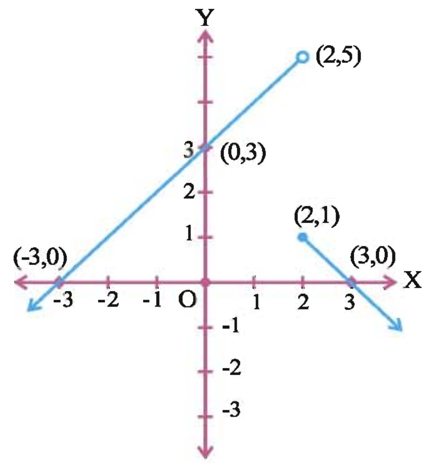

Figure 5.10

Solution : Let $a < 2$. So $f(x) = x + 3$ in some interval around a.

$$\lim_{x \to a} f(x) = \lim_{x \to a} (x + 3) = a + 3 = f(a)$$

$\therefore$ f is continuous for all $x \in R$, with $x < 2$.

Let $a > 2$. So $f(x) = 3 - x$ in some interval around a.

$\therefore$ $f(a) = 3 - a$

$$\lim_{x \to a} f(x) = \lim_{x \to a} (3 - x) = 3 - a = f(a)$$

$\therefore$ f is continuous for all $x \in R$, with $x > 2$.

Let $a = 2$. $\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (x + 3) = 5$ $\quad$ $(x < 2)$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (3 - x) = 1$$

$\therefore$ $\lim_{x \to 2} f(x)$ does not exist.

$\therefore$ f is continuous for all $x \in R$ except at $x = 2$.

[**Note :** Generally, f is continuous at all points where possibly formula for $f(x)$ changes or its graph is in transition stage.]

Example 10 : Find points of discontinuity of

$$f(x) = \begin{cases} x + 1 & x > 2 \\ 0 & x = 2 \\ 1 - x & x < 2 \end{cases}$$

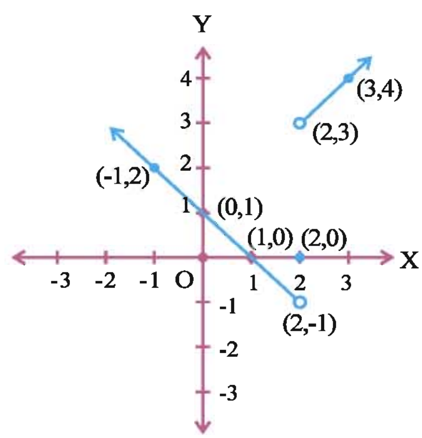

Figure 5.11

Solution : As per above note and a look at the graph of $y = f(x)$, it is clear that f is continuous at all $x \in R$ except at $x = 2$ possibly.

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (1 - x) = 1 - 2 = -1$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (x + 1) = 2 + 1 = 3$$

$\therefore$ $\lim_{x \to 2} f(x)$ does not exist.

$\therefore$ f is discontinuous at $x = 2$.

Example 11 : Prove that $f(x) = \begin{cases} x - 1 & x < 1 \\ 1 - x & x > 1 \end{cases}$ is continuous on $R - \{1\}$.

Solution : Let $a < 1$. So $f(a) = a - 1$.

For some $\delta > 0$, we can have $a + \delta < 1$.

Let $x \in (a - \delta, a + \delta)$. $f(x) = x - 1$

$\therefore \quad \lim_{x \to a} f(x) = \lim_{x \to a} (x - 1) = a - 1 = f(a)$

$\therefore \quad f$ is continuous for $a \in R$ with $a < 1$.

Let $a > 1$. So $f(a) = 1 - a$

For some $\delta > 0$, we can have $a - \delta > 1$

Let $x \in (a - \delta, a + \delta)$. Hence $x > 1$.

$\therefore \quad f(x) = 1 - x$

$\therefore \quad \lim_{x \to a+} f(x) = \lim_{x \to a+} (1 - x) = 1 - a = f(a)$

$\therefore \quad f$ is continuous for all $a \in R$ such that $a > 1$.

$\therefore \quad f$ is continuous on its domain.

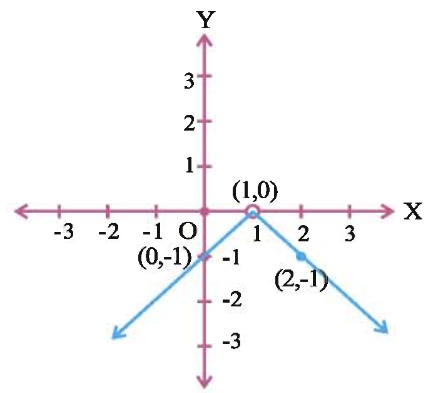

Figure 5.12

Example 12 : If $f(x) = \begin{cases} x - 1 & x < 1 \\ 0 & x = 1 \\ 1 - x & x > 1 \end{cases}$

Examine continuity of f.

Solution : As seen in example 11, f is continuous for all $x \in R, x \neq 1$.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x - 1) = 0, \quad \lim_{x \to 1+} f(x) = \lim_{x \to 1+} (1 - x) = 0$$

$\therefore \quad f(1) = 0$

$\therefore \quad f$ is continuous for $x = 1$.

$\therefore \quad f$ is continuous on R.

Note : Is not $f : R \to R, f(x) = -|x - 1|$?

Example 13 : If $f(x) = \begin{cases} x + 2 & x < 0 \\ 2 - x & x > 0 \\ k & x = 0 \end{cases}$

determine k so that f is continuous on R.

Solution : Looking at the graph and since $f(x) = 2 - x$ for $x > 0$ and $f(x) = x + 2$ for $x < 0$ are linear polynomials, f is continuous for all $x \in R - \{0\}$.

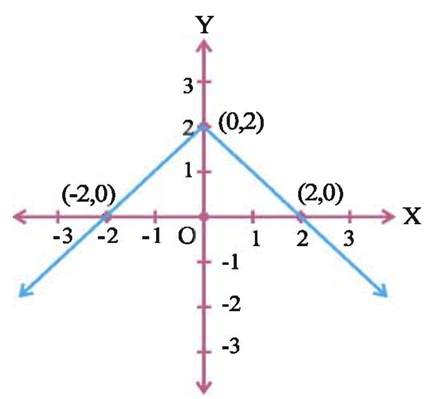

Figure 5.13

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x + 2) = 2$

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (2 - x) = 2$

$\therefore \quad \lim_{x \to 0} f(x) = 2$

In order that f is continuous at $x = 0$ also, $\lim_{x \to 0} f(x) = 2 = f(0)$ is necessary.

$\therefore \quad f(0) = k = 2$

$\therefore \quad$ If $k = 2$, f is continuous for all $x \in R$.

Example 14 : Prove that a polynomial function is continuous.

Solution : $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, $a_i \in R$ $(i = 0, 1, 2,..., n)$ $a_n \neq 0$ is a polynomial.

We know $\lim_{x \to a} x^n = a^n$

$\lim_{x \to a} a_i = a_i$ **(limit of a constant function)**

Also $\lim_{x \to a}(f_1(x) + f_2(x) + ... + f_n(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + ... + \lim_{x \to a} f_n(x)$

Now $\lim_{x \to a} f(x) = \lim_{x \to a} (a_n x^n + a_{n-1} x^{n-1} + ... + a_0)$

$= \lim_{x \to a} a_n \lim_{x \to a} x^n + \lim_{x \to a} a_{n-1} \lim_{x \to a} x^{n-1} + ... + \lim_{x \to a} a_0$

$= a_n a^n + a_{n-1} a^{n-1} + ... + a_0$

$= f(a)$

$\therefore$ A polynomial function is continuous for all $x \in R$.

Example 15 : Prove $f(x) = [x]$ is continuous at all $x \in R$ except at all integers.

Solution : $f(x) = \begin{cases} ... & .... \\ ... & .... \\ ... & .... \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ ... & .... \\ ... & .... \\ ... & .... \end{cases}$

$\therefore$ f is a constant function in any interval $(n, n + 1)$ where $n \in Z$.

$\therefore$ f is continuous in all intervals $(n, n + 1)$ i.e. at all $x \in R - Z$.

Now $f(x) = \begin{cases} n - 1 & n - 1 \le x < n \\ n & n \le x < n + 1 \end{cases}$

Let $x = n, n \in Z$

We can choose $\delta > 0$ such that $n - 1 < n - \delta < n$. **(In fact $0 < \delta < 1$)**

$\therefore$ $\lim_{x \to n-} f(x) = \lim_{x \to n-} n - 1 = n - 1$ **($x \in (n - \delta, n)$)**

Choose $\delta > 0$ so that $n < n + \delta < n + 1$. **($0 < \delta < 1$)**

$\therefore$ $\lim_{x \to n+} f(x) = \lim_{x \to n+} n = n$ **($x \in (n, n + \delta)$)**

$\therefore$ $\lim_{x \to n} f(x)$ does not exist. (See figure 5.1)

$\therefore$ f is discontinuous for all integers.

$\therefore$ $f(x) = [x]$ is continuous on $R - Z$ and discontinuous for all $n \in Z$.

Example 16 : Find k, if the following function is continuous at $x = 2$

$f(x) = \begin{cases} kx + 3 & x \le 2 \\ 7 & x > 2 \end{cases}$

Solution : $\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (kx + 3) = 2k + 3$

$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} 7 = 7$

$\therefore \quad \lim_{x \to 2} f(x)$ exists if $2k + 3 = 7$ i.e. $k = 2$

For $k = 2$, $f(2) = 2 \cdot 2 + 3 = 7$

$\therefore \quad \lim_{x \to 2} f(x) = 7 = f(2)$

$\therefore \quad f$ is continuous at $x = 2$, if $k = 2$.

Example 17 : Find a and b so that the following function is continuous.

$$f(x) = \begin{cases} 3 & x \leq 1 \\ ax + b & 1 < x < 3 \\ 7 & x \geq 3 \end{cases}$$

Solution : f is a constant function except for $x \in (1, 3)$

f is a linear polynomial in $(1, 3)$. So it is continuous function.

Hence, f is continuous for $x \in R - \{1, 3\}$ and in $(1, 3)$ except for possibly $x = 1$ and 3.

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (ax + b) = a + b, \quad \lim_{x \to 1-} f(x) = \lim_{x \to 1-} 3 = 3$

Since f is required to be continuous at $x = 1$, $\lim_{x \to 1} f(x)$ must exist.

$\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$

$\therefore \quad a + b = 3$ **(i)**

$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} (ax + b) = 3a + b, \quad \lim_{x \to 3+} f(x) = \lim_{x \to 3+} 7 = 7$

Since f is required to be continuous at $x = 3$, $\lim_{x \to 3} f(x)$ must exist.

$\lim_{x \to 3+} f(x) = \lim_{x \to 3-} f(x)$

$\therefore \quad 3a + b = 7$ **(ii)**

Solving (i) and (ii), $a = 2$, $b = 1$. Also $\lim_{x \to 1} f(x) = 3$, $\lim_{x \to 3} f(x) = 7$.

Now, $f(1) = 3$, $\lim_{x \to 1} f(x) = 3 = f(1)$

$f(3) = 7$, $\lim_{x \to 3} f(x) = 7 = f(3)$

$\therefore \quad$ If $a = 2$ and $b = 1$, f is continuous on R.

Example 18 : Find a and b, if following function is continuous at $x = 0$ and 1.

$$f(x) = \begin{cases} x + a & x < 0 \\ 2 & 0 \leq x < 1 \\ bx - 1 & 1 \leq x < 2 \end{cases}$$

Solution : $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x + a) = a$

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 2 = 2.$

Since f is continuous at $x = 0$, $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x)$

$\therefore \quad a = 2$. Also $f(0) = 2$

$\therefore \quad \lim_{x \to 0} f(x) = 2 = f(0)$

$\therefore$ Taking $a = 2$, f is continuous at $x = 0$.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 2 = 2$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (bx - 1) = b - 1$$

Since, f is continuous at $x = 1$, $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$

$\therefore$ $b - 1 = 2$

$\therefore$ $b = 3$

Also, $f(1) = b - 1 = 3 - 1 = 2$

$\therefore$ $\lim_{x \to 1} f(x) = 2 = f(1)$

$\therefore$ Taking $a = 2$ and $b = 3$, f is continuous at $x = 0$ and $x = 1$.

5.3 Algebra of continuous functions

The concept of continuity is formulated in terms of limit. Hence, just like working rules of limit, we can have working rules for continuity of $f \pm g$, $f \times g$, $\frac{f}{g}$, etc.

Theorem 5.1 : Let f and g be continuous at $x = c$ and $c \in (a, b)$ for some interval (a, b).

Then (1) $f + g$ is continuous at $x = c$.

(2) kf is continuous at $x = c$. $k \in R$

(3) $f - g$ is continuous at $x = c$.

(4) $f \times g$ is continuous at $x = c$.

(5) $\frac{k}{g}$ is continuous at $x = c$ if $g(c) \neq 0$. $k \in R$

(6) $\frac{f}{g}$ is continuous at $x = c$ if $g(c) \neq 0$

$\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$ as f, g are continuous at $x = c$.

(1) $\lim_{x \to c} (f + g)(x) = \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

$= f(c) + g(c)$

$= (f + g)(c)$

$\therefore$ $f + g$ is continuous at $x = c$.

(2) $\lim_{x \to c} (kf)(x) = \lim_{x \to c} kf(x)$

$= \lim_{x \to c} k \lim_{x \to c} f(x)$

$= kf(c)$

$= (kf)(c)$

$\therefore$ kf is continuous at $x = c$.

(3) If $k = -1$, $-g$ is continuous at $x = c$ as g is continuous.

$\therefore$ $f + (-g) = f - g$ is continuous at $x = c$.

(4) $$\lim_{x \to c} (f \times g)(x) = \lim_{x \to c} f(x)g(x)$$

$$= \lim_{x \to c} f(x) \lim_{x \to c} g(x)$$

$$= f(c)\, g(c)$$

$$= (f \times g)(c)$$

$\therefore$ $f \times g$ is continuous at $x = c$.

(5) $$\lim_{x \to c} \left(\frac{k}{g}\right)(x) = \frac{\lim\limits_{x \to c} k}{\lim\limits_{x \to c} g(x)} \qquad (g(x) \neq 0)$$

$$= \frac{k}{g(c)} \qquad (g(c) \neq 0)$$

$\therefore$ $\frac{k}{g}$ is continuous at $x = c$.

(6) $$\left(\frac{f}{g}\right)(x) = \left(f \times \frac{1}{g}\right)(x)$$

Taking $k = 1$ in (5), $\frac{1}{g}$ is continuous at $x = c$.

$\left(f \times \frac{1}{g}\right) = \frac{f}{g}$ is continuous at $x = c$.

or

$$\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \lim_{x \to c} \frac{f(x)}{g(x)}$$

$$= \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$$

$$= \frac{f(c)}{g(c)} \qquad (g(c) \neq 0)$$

$$= \left(\frac{f}{g}\right)(c)$$

$\therefore$ $\frac{f}{g}$ is continuous at $x = c$.

Some Important Results :

(1) A rational function is continuous on its domain.

$h(x) = \frac{p(x)}{q(x)}$ is a rational function, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$

$$\lim_{x \to a} h(x) = \lim_{x \to a} \frac{p(x)}{q(x)}$$

$$= \frac{\lim\limits_{x \to a} p(x)}{\lim\limits_{x \to a} q(x)}$$

$$= \frac{p(a)}{q(a)} \qquad (q(a) \neq 0)$$

$$= h(a)$$

$\therefore$ h is a continuous on its domain.

(2) *sine* function is continuous on R.

We assume following results studied last year

$\lim\limits_{x \to 0} sinx = 0, \quad \lim\limits_{x \to 0} cosx = 1$

Let $a \in$ R. Let $x = a + h$, so that as $x \to a$, $h \to 0$

$$\lim_{x \to a} sinx = \lim_{h \to 0} sin\,(a + h)$$

$$= \lim_{h \to 0} (sina\ cosh + cosa\ sinh)$$

$$= sina \lim_{h \to 0} cosh + cosa \lim_{h \to 0} sinh$$

$$= sina \cdot 1 + cosa \cdot 0$$

$$= sina$$

$\therefore$ $\lim\limits_{x \to a} sinx = sina$

$\therefore$ *sine* function is continuous for all $x \in$ R.

(3) *cosine* function is continuous on R.

Let $a \in$ R. Let $x = a + h$. As $x \to a$, $h \to 0$

$$\lim_{x \to a} cosx = \lim_{h \to 0} cos(a + h)$$

$$= \lim_{h \to 0} (cosa\ cosh - sina\ sinh)$$

$$= cosa \lim_{h \to 0} cosh - sina \lim_{h \to 0} sinh$$

$$= cosa \cdot 1 - sina \cdot 0$$

$$= cosa$$

$\therefore$ $\lim\limits_{x \to a} cosx = cosa$

$\therefore$ *cosine* function is continuous for all $x \in$ R.

(4) *tan* function is continuous :

$tanx = \frac{sinx}{cosx}, x \in \text{R} - \left\{(2k - 1)\frac{\pi}{2} \,\middle|\, k \in \text{Z}\right\}$

sine is continuous for $x \in$ R.

cosine is continuous for $x \in$ R.

$cosx = 0 \Leftrightarrow x \in \text{R} - \left\{(2k - 1)\frac{\pi}{2} \,\middle|\, k \in \text{Z}\right\}$

$\therefore$ By working rule of $\frac{f}{g}$ for continuous functions f and g, *tan* function is continuous on its domain.

(5) Continuity of Composite Function :

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to (e, f)$ be two functions, so that gof is defined.

If f is continuous at $x_1 \in (a, b)$ and g is continuous at $f(x_1) \in (c, d)$, then gof is continuous at $x_1 \in (a, b)$.

According to the rule of limit of composite functions (std XI, semester II).

$$\lim_{x \to x_1} g(f(x)) = g(\lim_{x \to x_1} (f(x)) = g(f(x_1))$$

$\therefore$ gof is continuous at $x = x_1$.

Example 19 : Prove that $x - [x]$ is discontinuous for all $n \in Z$.

Solution : $f(x) = \begin{cases} .... & .... \\ .... & .... \\ x & 0 \le x < 1 \\ x - 1 & 1 \le x < 2 \\ x - 2 & 2 \le x < 3 \\ .... & .... \\ .... & .... \end{cases}$

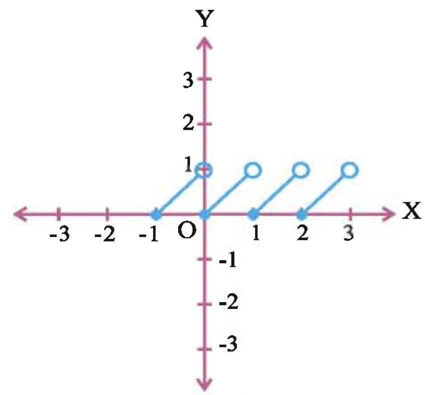

Figure 5.14

For any $n \in Z$

$$\lim_{x \to n-} f(x) = \lim_{x \to n-} (x - [x])$$

$$= \lim_{x \to n-} (x - (n - 1)) \quad \text{(For } 0 < \delta < 1, x \in (n - \delta, n))$$

$$= n - (n - 1)$$

$$= 1$$

and $f(n) = n - [n] = n - n = 0$

$\therefore \lim_{x \to n-} f(x) \neq f(n) \quad \forall n \in Z$

$\therefore$ $f(x) = x - [x]$ is not continuous for $n \in Z$.

Note : On intervals $(0, 1)$, $(1, 2)$,... etc. $f(x) = x - [x]$ is continuous. Let if possible, $x - [x]$ be continuous for $n \in Z$. $g(x) = x$ is continuous on R.

$\therefore$ $f(x) = x - [x]$ and $g(x) = x$ both are continuous on R.

$\therefore$ $g(x) - f(x) = x - (x - [x]) = [x]$ is also continuous on R. But $[x]$ is discontinuous for $n \in Z$. So $f(x) = x - [x]$ is not continuous for $n \in Z$.

Example 20 : Prove $sin\,|x|$ is continuous on R.

Solution : $f : R \to R$, $f(x) = |x|$ and $g : R \to R$, $g(x) = sinx$ are continuous.

$\therefore$ $gof : R \to R$, $(gof)(x) = g(f(x)) = g(|x|) = sin\,|x|$ is continuous for all $x \in R$.

Example 21 : Prove $f : R \to R$, $f(x) = |\,1 - x + |x|\,|$ is continuous.

Solution : $g(x) = 1 - x$ and $h(x) = |x|$ are continuous on R.

$\therefore$ $g(x) + h(x) = 1 - x + |x|$ is continuous.

$\therefore$ $f(x) = ho((g + h)(x)) = h((g + h)(x)) = |\,1 - x + |x|\,|$ is continuous as h, g are continuous on R.

Example 22 : Prove $cos(x^3)$ is continuous on R.

Solution : $f : \mathrm{R} \to \mathrm{R}, f(x) = x^3, g : \mathrm{R} \to \mathrm{R}, g(x) = cosx$ are continuous.

$\therefore$ $gof : \mathrm{R} \to \mathrm{R}, (gof)(x) = g(f(x)) = g(x^3) = cosx^3$ is continuous.

Example 23 : $f(x) = \begin{cases} \dfrac{kcosx}{\pi - 2x} & x \neq \dfrac{\pi}{2} \\ k^2 & x = \dfrac{\pi}{2} \end{cases}$

Can you find k so that f is continuous at $x = \frac{\pi}{2}$?

Solution : $\lim\limits_{x \to \frac{\pi}{2}} f(x) = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{kcosx}{2\left(\frac{\pi}{2} - x\right)} = \lim\limits_{\alpha \to 0} \dfrac{k \sin \alpha}{2\alpha} = \dfrac{k}{2}$ $\left(\alpha = \frac{\pi}{2} - x\right)$

$f\left(\frac{\pi}{2}\right) = k^2$

Since f is continuous at $x = \frac{\pi}{2}$, $\lim\limits_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$\therefore$ $\frac{k}{2} = k^2$

$\therefore$ $k = \frac{1}{2}$ or 0

[**Note :** For $k = 0$, $f(x) = 0$ for all $x \in \mathrm{R}$.]

Example 24 : $f(x) = \begin{cases} \dfrac{sinx}{|x|} & x \neq 0 \\ k & x = 0 \end{cases}$

Can you find k so that f is continuous at $x = 0$?

Solution : $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} \dfrac{sinx}{|x|} = \lim\limits_{x \to 0+} \dfrac{sinx}{x} = 1$

$\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} \dfrac{sinx}{|x|} = \lim\limits_{x \to 0-} \dfrac{sinx}{-x} = -1$

$\therefore$ $\lim\limits_{x \to 0} f(x)$ does not exist.

$\therefore$ f cannot be continuous for $x = 0$, for any value of $k \in \mathrm{R}$.

Example 25 : $f(x) = \begin{cases} \dfrac{sin4x}{9x} & x \neq 0 \\ k^2 & x = 0 \end{cases}$

Find k, if f is continuous for $x = 0$.

Solution : $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{sin4x}{9x}$

$= \lim\limits_{x \to 0} \dfrac{sin4x}{4x} \dfrac{4}{9}$

$= \dfrac{4}{9}$

$\therefore \quad f(0) = k^2$

$\therefore \quad k^2 = \frac{4}{9}$ for f to be continuous at $x = 0$.

$\therefore \quad k = \pm\frac{2}{3}$ for f to be continuous at $x = 0$.

Exercise 5.1

1. Prove *cot*, *cosec* and *sec* are continuous on their domains.
2. Prove ceiling function $f(x) = \lceil x \rceil$ is discontinuous for all $n \in Z$.
3. Prove signum function is discontinuous at $x = 0$.

Discuss continuity of following functions : (4 to 12)

4. $f(x) = \begin{cases} x + 3 & x \geq 2 \\ 3 - x & x < 2 \end{cases}$

5. $f(x) = \begin{cases} x^2 & x \geq 0 \\ x & x < 0 \end{cases}$

6. $f(x) = \begin{cases} 2x + 3 & x < 1 \\ 5 & x = 1 \\ 3x + 2 & x > 1 \end{cases}$

7. $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$

8. $f(x) = \begin{cases} \frac{\tan x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

9. $f(x) = \begin{cases} 2x - 3 & x < 0 \\ 2 & x = 0 \\ 3x - 2 & x > 0 \end{cases}$

10. $f(x) = \begin{cases} \frac{\sin x}{3x} & x \neq 0 \\ \frac{2}{3} & x = 0 \end{cases}$

11. $f(x) = \begin{cases} \frac{2x+3}{3x+2} & x > 0 \\ \frac{\sin 3x}{2x} & x < 0 \\ \frac{3}{2} & x = 0 \end{cases}$

12. $f(x) = \begin{cases} \frac{x^2-1}{x^2+1} & x > 0 \\ \frac{\sin x}{|x|} & x < 0 \\ -1 & x = 0 \end{cases}$

Determine k, if following functions are continuous at given values of x : (13 to 16)

13. $f(x) = \begin{cases} \frac{\tan kx}{3x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ **(at $x = 0$)**

14. $f(x) = \begin{cases} \frac{\sin 5x}{kx} & x \neq 0 \\ 1 & x = 0 \end{cases}$ **(at $x = 0$)**

15. $f(x) = \begin{cases} \frac{(x+1)\tan(x-1)}{\sin(x^2-1)} & x \neq 1 \\ k & x = 1 \end{cases}$ **(at $x = 1$)**

16. $f(x) = \begin{cases} 2x^2 + k & x < 0 \\ x^2 - 2k & x \geq 0 \end{cases}$ **(at $x = 0$)**

Find a and b if f is continuous :

17. $f(x) = \begin{cases} 2x + 3 & 1 < x < 2 \\ ax + b & 2 \le x < 3 \\ 3x + 2 & 3 \le x \le 4 \end{cases}$ **(at $x = 2$ and $x = 3$)**

18. Prove $\sin^2 x - \cos^2 x$ is continuous on R.

19. Prove $\sin 2x \cos 3x$ is continuous on R.

20. Prove $\sin |x|$ is continuous on R.

21. Prove $|\sin x|$ is continuous on R.

22. Prove $\sin^3 x$ and $\sin x^3$ are continuous on R.

23. Prove $\cos x^n$ is continuous on R. ($n \in$ N)

24. Prove $\cos^n x$ is continuous on R. ($n \in$ N)

25. $f(x) = \begin{cases} \sin x - \cos x & x \ne 0 \\ -1 & x = 0 \end{cases}$

Prove f is continuous at $x = 0$.

26. $f(x) = \begin{cases} |\sin x - \cos x| & x \ne 0 \\ -1 & x = 0 \end{cases}$

Is f is continuous at $x = 0$?

27. $f(x) = \begin{cases} \dfrac{\sin x - \cos x}{x - \frac{\pi}{4}} & x \ne \frac{\pi}{4} \\ k & x = \frac{\pi}{4} \end{cases}$

If f is continuous at $x = \frac{\pi}{4}$, find k.

28. $f(x) = \begin{cases} \dfrac{x^n - 2^n}{x - 2} & x \ne 2 \\ 80 & x = 2 \end{cases}$

If f is continuous at $x = 2$, find n.

*

5.4 Exponential and Logarithmic Functions

The function $f(x) = x^n$ is used in polynomial functions and rational functions.

Let $f_n(x) = x^n$.

$f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$,..... etc.

Let us draw the graphs.

For $f_2(x)$,

x	1	2	3	4	5	–1	–2	–3
$f_2(x)$	1	4	9	16	25	1	4	9

For $f_3(x)$,

x	1	2	3	4	5	–1	–2	–3
$f_3(x)$	1	8	27	64	125	–1	–8	–27

As x increasees, $f_n(x)$ increases. For a fixed increment in x, where $x > 1$, the increment in $f_n(x)$ increases as n increases. For example if x increases from 2 to 3, $f_{10}(2) = 2^{10}$, $f_{10}(3) = 3^{10}$, $f_{20}(2) = 2^{20}$, $f_{20}(3) = 3^{20}$.

Obviously $3^{20} - 2^{20} > 3^{10} - 2^{10}$.

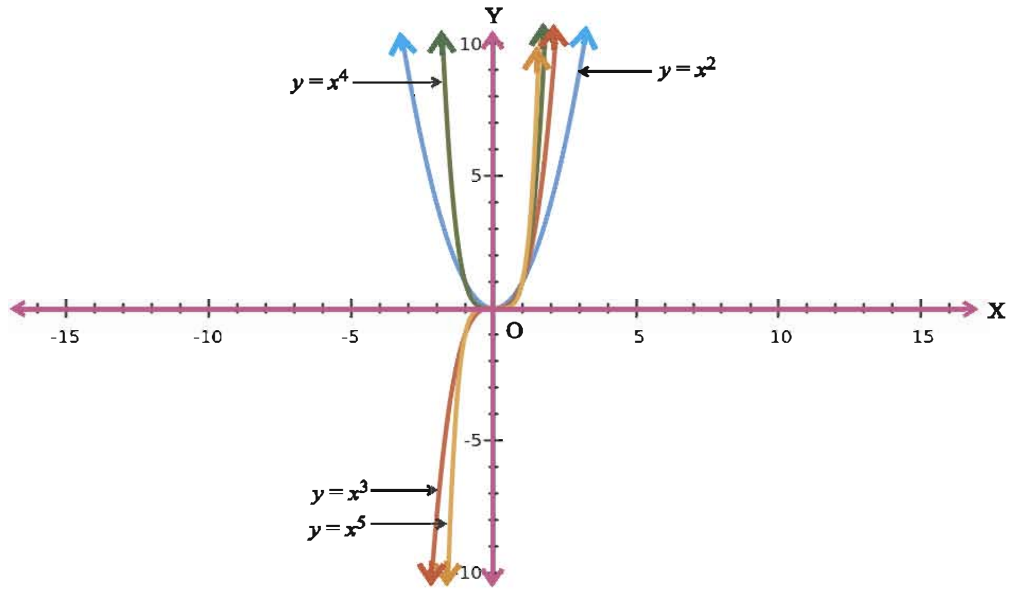

Figure 5.15

Now we consider 'common exponential' function $f(x) = 10^x$. This function increases faster than any $f_n(x)$. Let $x = 10^2$.

Now, $f_{100}(x) = x^{100} = (10^2)^{100} = 10^{200}$, $f(x) = 10^{10^2} = 10^{100}$

For $x = 10^3$, $f_{100}(x) = x^{100} = 10^{300}$, $f(x) = 10^{10^3} = 10^{1000}$

For $x = 10^4$, $f_{100}(x) = (10^4)^{100} = 10^{400}$, $f(x) = 10^{10^4} = 10^{10000}$

Obviously, if $x > 10^3$, $f(x)$ increases much faster than $f_{100}(x)$.

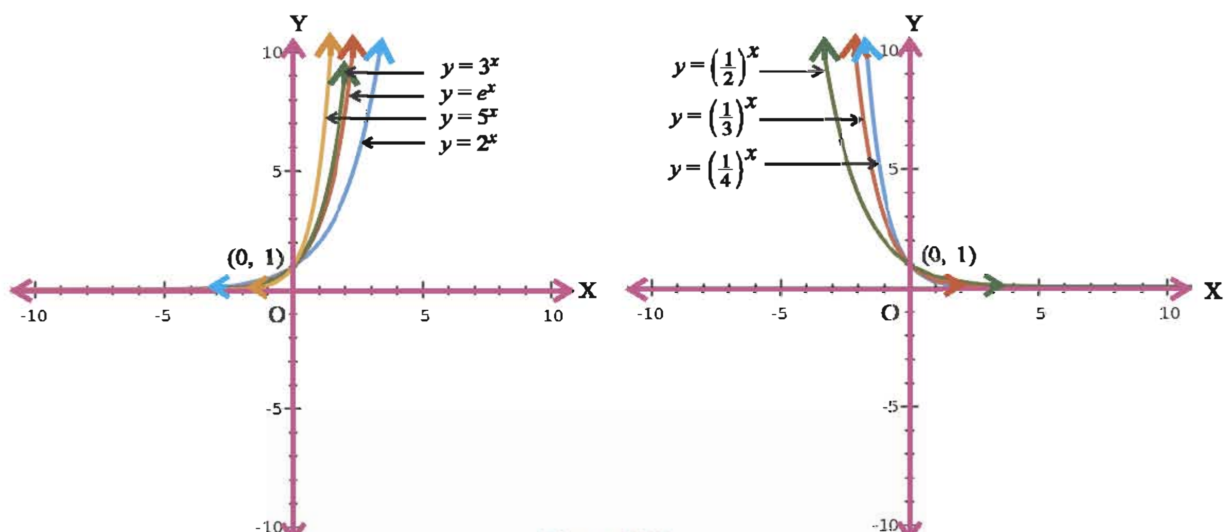

Figure 5.16

Exponential Function : $f(x) = a^x$, $a \in R^+$, $x \in R$ **is called an exponential function.**

(1) If $a > 1$, $f(x)$ increases as x increases.

If $a < 1$, $f(x)$ decreases as x increases.

(2) The graph of $f(x)$ passes through (0, 1) for any $a \in R^+$.

(3) If $a \neq 1$, the function is one-one and onto.

(4) Its range is R^+.

(5) If a becomes larger, the graph of $f(x)$ leans towards Y-axis for $a > 1$.

(6) As x becomes negative and decreases, the graph of $f(x)$ approaches X-axis but does not intersect X-axis.

Laws of indices for real numbers :

(1) $a^x a^y = a^{x+y}$ (2) $\frac{a^x}{a^y} = a^{x-y}$

(3) $(a^x)^y = a^{xy}$ (4) $(ab)^x = a^x b^x$ $a, b \in R^+$ $x, y \in R$

(This content is only for link to the discussion that follows and this is not from examination view point).

The constant e : Limit of a sequence : Just like functions, some sequences also approach a 'limit'.

The sequence $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{100}, \frac{1}{n}$.... has terms nearing 0.

We say $\lim\limits_{n \to \infty} \frac{1}{n} = 0$

We do not formally define limit of a sequence. We accept following results.

(1) $\lim\limits_{n \to \infty} \frac{1}{n} = 0$. $(n \in N)$ We also assume $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ $(x \in R)$

(2) $\lim\limits_{n \to \infty} r^n = 0$ $|r| < 1$

For example if $r = \frac{1}{2}$, we have the sequence, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$,.. and $\left(\frac{1}{2}\right)^n$ approaches 0 as n becomes larger and larger.

Consider the sequence

$$\left(1+\frac{1}{n}\right)^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + ... + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{n(n-1)}{2!\,n^2} + \frac{n(n-1)(n-2)}{3!\,n^3} + ... + \frac{n(n-1).....1}{n!\,n^n}$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + ... + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)....\left(1-\frac{n-1}{n}\right)}{n!}$$

$1 - \frac{1}{n}, 1 - \frac{2}{n}, 1 - \frac{3}{n}$ are all less than 1 and hence their products wherever occuring are less than 1.

$$\therefore \quad \left(1+\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ... + \frac{1}{n!} \qquad (n > 1)$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{n-1}} \qquad (2^{n-1} < n!)$$

$$\therefore \quad \left(1+\frac{1}{n}\right)^n < 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \qquad \text{(Geometric Progression)}$$

$$\therefore \quad \left(1+\frac{1}{n}\right)^n < 1 + 2\left(1-\left(\frac{1}{2}\right)^n\right) = 3 - 2\left(\frac{1}{2}\right)^n < 3 \qquad \text{(i)}$$

Obviously $\left(1+\frac{1}{n}\right)^n > 2$ **($n > 1$) (ii)**

We assume sequence $\left(1+\frac{1}{n}\right)^n$ has a limit called e and by (i) and (ii) above $2 < e < 3$.

Thus e is a definite constant satisfying $2 < e < 3$. It is called Napier's constant.

Approximatly $e = 2.71828183$

$$\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e$$

We can prove but we will not prove $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e$ or replacing $\frac{1}{x}$ by x, $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

Logarithmic Function :

We know **exponential function $f : R \to R^+$, $f(x) = a^x$ ($a \in R^+ - \{1\}$) is one-one and onto.**

Its inverse function $g : R^+ \to R$ is called logarithmic function. So if $y = f(x) = a^x$, then $x = g(y) = \log_a y$

This function is denoted as $g = \log_a$

If $y = a^x$, then $x = \log_a y$

We know for inverse functions, $f : A \to B$ and $g : B \to A$, $(fog)(y) = y$, $y \in B$ and $(gof)(x) = x$, $x \in A$

Now $f(g(y)) = y$

$\therefore \quad f(\log_a y) = y$

$\therefore \quad a^{\log_a y} = y$

or in other words, **$a^{\log_a x} = x$ for $x \in R^+$**

If $a = 10$, we get what is called common logarithm. i.e. $\log_{10} x$

Thus, $f : R \to R^+$, $f(x) = 10^x$ has inverse $\log_{10} : R^+ \to R$, $g(x) = \log_{10} x$

If $a = e$, we get natural logarithm and it is denoted by $\ln_e x$. But unless otherwise stated, we will write $\ln x$ as $\log_e x$ or simply $\log x$.

(1) log has domain R^+ and range R. Hence, logarithm of only positive number can be obtained and log x is a real number if $x \in R^+$.

(2) $a^0 = 1$. Hence $\log_a 1 = 0$

Hence $\log_e 1 = 0$, $\log_{10} 1 = 0$

(3) $a^1 = a$. Hence $\log_a a = 1$

$\log_e e = 1$, $\log_{10} 10 = 1$

$e^{\log_e x} = x$ as $a^{\log_a x} = x$ for $a \in R^+ - \{1\}$

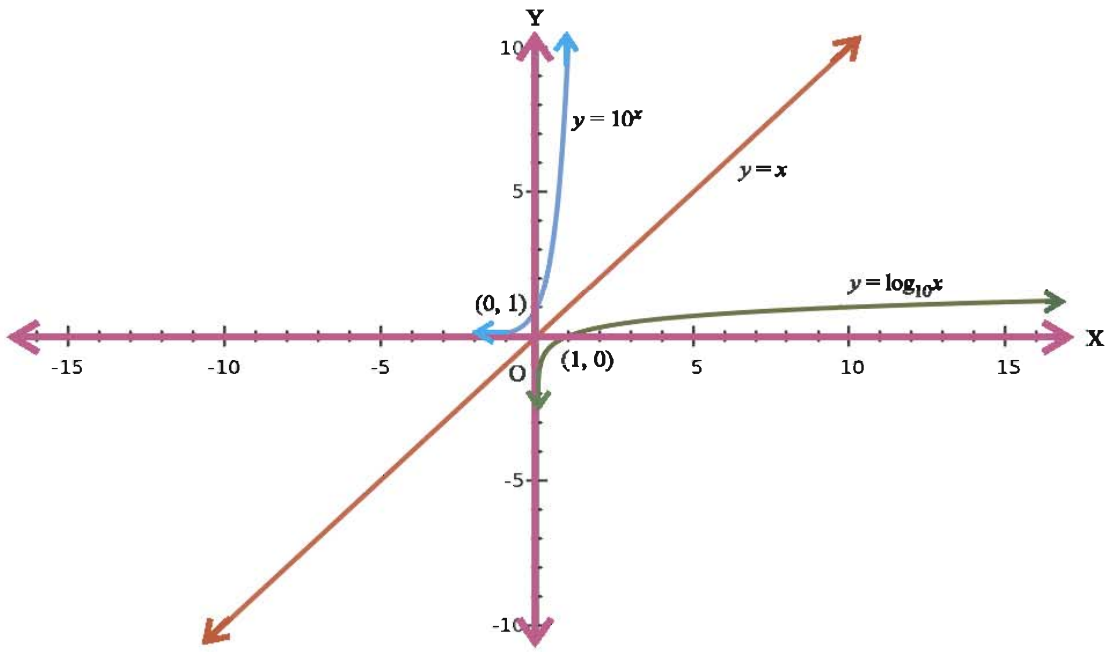

Figure 5.17

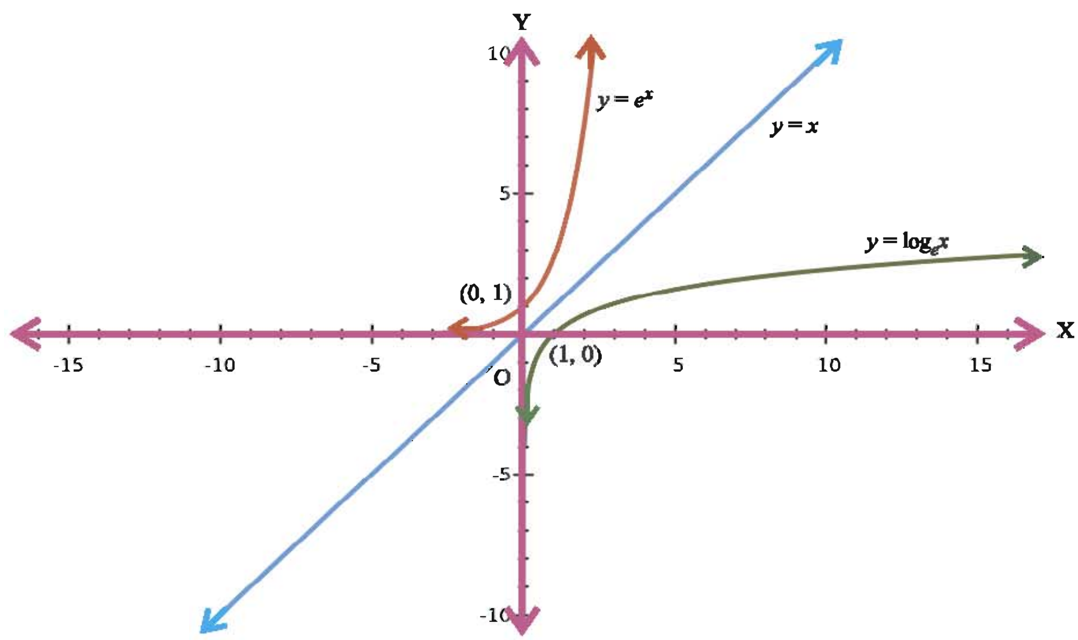

Figure 5.18

We can see that graphs of $f(x) = \log_e x$ and $f(x) = e^x$ are mirror images of each other in the line $y = x$.

(1) (1, 0) is on the graph of log function.

(2) For $a > 1$, it is increasing.

For $0 < a < 1$, it is decreasing.

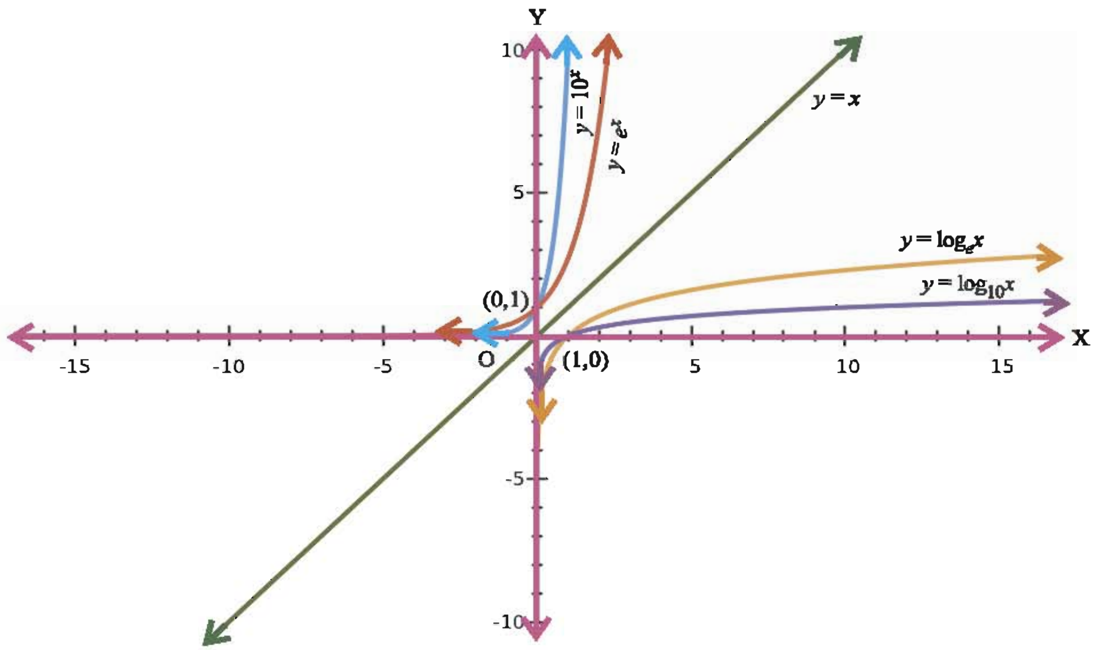

Figure 5.19

Some rules for logarithm :

(1) $\log_a mn = \log_a m + \log_a n$ $\qquad (m, n \in R^+, a \in R^+ - \{1\})$

Let $\log_a m = x$, $\log_a n = y$

$\therefore \quad m = a^x, n = a^y$

$\therefore \quad mn = a^x a^y = a^{x+y}$

$\therefore \quad \log_a mn = x + y = \log_a m + \log_a n$

(2) $\log_a \frac{m}{n} = \log_a m - \log_a n$ $\qquad (m, n \in R^+, a \in R^+ - \{1\})$

Proof is similar as in (1)

(3) $\log_a x^n = n\log_a x$ $\qquad (x \in R^+, n \in Z, a \in R^+ - \{1\})$

Let $\log_a x = y$

$\therefore \quad x = a^y$

$\therefore \quad x^n = (a^y)^n = a^{ny}$

$\therefore \quad \log_a x^n = ny$

$\therefore \quad \log_a x^n = n\log_a x$

(4) Change of Basis Rule : $\log_a b = \dfrac{\log_c b}{\log_c a}$ **($b \in R^+$, $a, c \in R^+ - \{1\}$)**

Let $\log_a b = x$, $\log_c a = y$

$\therefore \quad b = a^x, \; a = c^y$

$\therefore \quad b = (c^y)^x = c^{xy}$

$\therefore \quad \log_c b = xy = \log_a b \times \log_c a$

$\therefore \quad \log_a b = \dfrac{\log_c b}{\log_c a}$ **(since $a \neq 1$, $\log_c a \neq 0$)**

Also $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = \lim\limits_{x \to 0} \dfrac{1}{x} \log(1+x)$

$= \lim\limits_{x \to 0} \log (1+x)^{\frac{1}{x}}$

$= \log \left(\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}} \right)$ **(log is continuous)**

$= \log_e e$

$= 1$

$\therefore \quad \lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1$

5.5 Differentiation

We have learnt the concept of differentiation last year. Let us remember.

If $f : (a, b) \to R$ is a function and if $c \in (a, b)$ and h is so small that $c + h \in (a, b)$, then $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$, if it exists, is called the derivative of f at c and is denoted by $f'(c)$ or $\left[\dfrac{d}{dx} f(x)\right]_{x = c}$ or $\left(\dfrac{dy}{dx}\right)_{x = c}$ where $y = f(x)$. If the derivative of f exists at $x = c$, we say f is differentiable at $x = c$. $\dfrac{dy}{dx}$ is also denoted by y_1.

If f is differentiable for all x in a set A, (A $\neq \emptyset$), we say f is differentiable in A.

f is differentiable at $c \in (a, b)$ means $\lim\limits_{h \to 0+} \dfrac{f(c+h) - f(c)}{h}$ and $\lim\limits_{h \to 0-} \dfrac{f(c+h) - f(c)}{h}$ both exist and are equal.

Let f be defined on $[a, b]$. f is differentiable in $[a, b]$ means

(1) **f is differentiable in (a, b)**

(2) $\lim\limits_{h \to 0+} \dfrac{f(a+h) - f(a)}{h}$ **exists.**

We call this limit right-hand derivative of f at $x = a$ and write $f'(a+)$.

(3) $\lim\limits_{h \to 0-} \dfrac{f(b+h) - f(b)}{h}$ **exists,**

We call this left-hand derivative of f at $x = b$ and denote it by $f'(b-)$.

We also assume following working rules and standard forms.

If f and g are differentiable at x,

(1) $f \pm g$ is differentiable at x and $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

(2) $f \times g$ is differentiable at x and $\frac{d}{dx}f(x)\,g(x)) = f(x)\,\frac{d}{dx}g(x) + g(x)\,\frac{d}{dx}f(x)$

(3) $\frac{f}{g}$ is differentiable at x if $g(x) \neq 0$ and $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

(4) $\frac{d}{dx}x^n = nx^{n-1}$ $\quad n \in R, x \in R^+$

(5) $\frac{d}{dx}\sin x = \cos x$ $\quad x \in R$

(6) $\frac{d}{dx}\cos x = -\sin x$ $\quad x \in R$

(7) $\frac{d}{dx}\tan x = \sec^2 x$ $\quad x \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$

(8) $\frac{d}{dx}\sec x = \sec x \tan x$ $\quad x \in R - \left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$

(9) $\frac{d}{dx}\cot x = -\text{cosec}^2 x$ $\quad x \in R - \{k\pi \mid k \in Z\}$

(10) $\frac{d}{dx}\text{cosec}\,x = -\text{cosec}\,x \cot x$ $\quad x \in R - \{k\pi \mid k \in Z\}$

Now we prove a theorem.

Theorem 5.2 : If f is differentiable at $x = c$, it is continuous at $x = c$. $c \in (a, b)$

Proof : Let f be differentiable at $x = c$.

$\therefore \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

Now $f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c)$ for $x \neq c$.

$$\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c}\frac{f(x) - f(c)}{x - c}\ \lim_{x \to c}(x - c)$$

(because f is differentiable, $\lim_{x \to c}\frac{f(x) - f(c)}{x - c}$ exists)

$$= f'(c) \cdot 0 = 0$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(f(x) - f(c) + f(c))$$

$$= \lim_{x \to c}(f(x) - f(c)) + \lim_{x \to c} f(c) \qquad \textbf{(both the limits exist)}$$

$$= 0 + f(c)$$

$$= f(c)$$

$\therefore$ f is continuous at $x = c$.

But a continuous function may not be differentiable.

Consider $f(x) = |x|$

$$\lim_{x \to 0+} |x| = \lim_{x \to 0+} x = 0, \quad \lim_{x \to 0-} |x| = \lim_{x \to 0-} (-x) = 0, \; f(0) = |0| = 0$$

$\therefore$ f is continuous at $x = 0$.

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = 1$$

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

$\therefore$ $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist.

$\therefore$ $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Can we explain the situation ?

We had seen that $f'(c)$ is the slope of tangent to $y = f(x)$ at $x = c$.

See that the graph of $f(x) = |x|$ consists of two rays meeting at (0, 0) and does not have a tangent at (0, 0). It has a 'corner'.

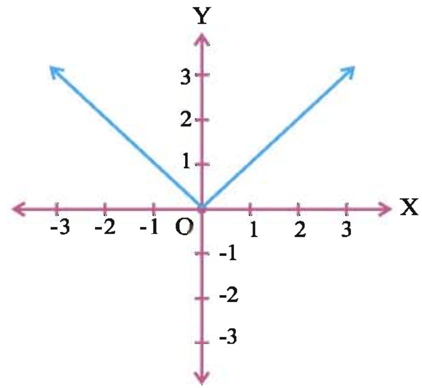

Figure 5.20

When can a function fail to have a derivative ?

(1) It is discontinuous at that point. (Fig. 5.21)

(2) The tangent is vertical at $x = c$. (Fig. 5.22)

(3) There is no tangent at $x = c$. (Fig. 5.23)

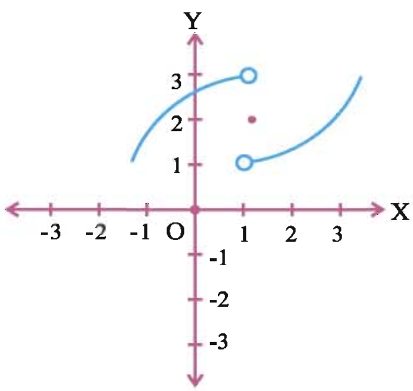

Figure 5.21

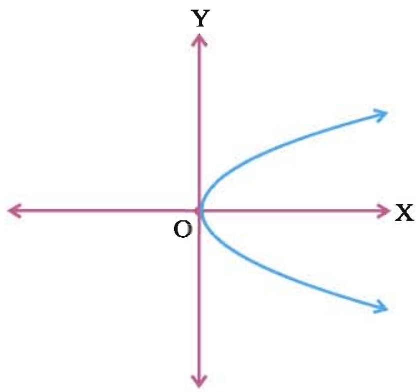

Figure 5.22

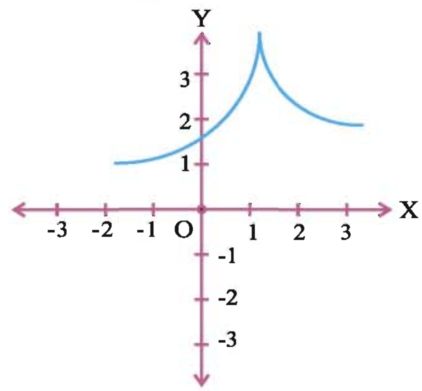

Figure 5.23

Exercise 5.2

1. Prove that $f(x) = |x - 1| + |x - 2| + |x - 3|$ is continuous on R but not differentiable at $x = 1$, 2 and 3 only.

2. Prove $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous but not differentiable at $x = 0$.

3. For $f(x) = \begin{cases} x^2 sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove $f'(0) = 0$. Deduce f is continuous at $x = 0$.

4. Find $f'(x)$ for (1) $f(x) = sin^2x$, (2) $f(x) = tan^2x$, (3) $f(x) = x^4$, (4) $f(x) = cos^4x$

*

5.6 Chain rule or Derivative of a Composite Function

We have seen how to find the derivative of sin^2x or tan^3x using product rule or the derivative of $sin2x$ or $cos2x$ using formulae from trigonometry like $sin2x = 2sinx\ cosx$, $cos2x = cos^2x - sin^2x$ along with product rule.

But they were simple cases. Suppose we want to find the derivative of $tan^5(x^2 - x + 1)$. It is not so easy.

Let us take an example.

Let $f(x) = (2x + 1)^4$

$= 16x^4 + 32x^3 + 24x^2 + 8x + 1$

$f'(x) = 64x^3 + 96x^2 + 48x + 8$

$= 8(8x^3 + 12x^2 + 6x + 1)$

$= 8(2x + 1)^3$

$= 2 \cdot 4\,(2x + 1)^3$

Let $g(t) = t^4$ and $t = h(x) = 2x + 1$. So, $g(h(x)) = g(2x + 1) = (2x + 1)^4 = f(x)$

$\therefore$ $f(x) = g(h(x))$

Now $g'(t) = 4t^3$ and $\frac{dt}{dx} = h'(x) = 2$

$f'(x) = 8(2x + 1)^3 = 4(2x + 1)^3 \cdot 2$

$= 4t^3 \cdot 2 = g'(t)\frac{dt}{dx} = g'(t)\ h'(x)$

So, $\frac{d}{dx}f(x) = \frac{d}{dx}g(h(x)) = g'(t)\,h'(x) = g'(h(x))\ h'(x)$

Here, we have expressed $f(x)$ as a composite function of two functions $g(t) = t^4$ and $h(x) = 2x + 1$ whose derivative can be found out in a very simple manner and $f'(x)$ can be calculated in a simple way.

Let us make it formal.

Chain rule : $f : (a, b) \to (c, d)$ is differentiable at x and $g : (c, d) \to (e, f)$ is differentiable at $f(x)$ are two differentiable functions.

Now, $(gof)(x) = g(f(x))$

Then $(gof)'(x) = g'(f(x))\ f'(x)$

In other words let $h(x) = (gof)(x) = g(f(x))$. Let $f(x) = t$

Then $h'(x) = (gof)'(x) = g'(f(x))\ f'(x)$

$= g'(t)\ f'(x)$

$\therefore$ $\frac{d}{dx}g(f(x)) = \frac{d}{dt}g(t)\frac{d}{dx}f(x)$. where $t = f(x)$

Thus, $\frac{d}{dx} g(f(x)) = \frac{du}{dt} \frac{dt}{dx}$. where $u = g(t)$ and $t = f(x)$.

Hence $\frac{du}{dx} = \frac{du}{dt} \frac{dt}{dx}$ where $u = g(t)$ and $t = f(x)$ and $u = g(f(x))$

Thus if u is a function of t and t is a function of x. Then u is a composite function of x and

$$\frac{du}{dx} = \frac{du}{dt} \frac{dt}{dx}$$

This rule is called chain rule.

Continuing $\frac{du}{dx} = \frac{du}{dt} \frac{dt}{ds} \frac{ds}{dv} \frac{dv}{dx}$

Here u is a function of t, t is a function of s, s is a function of v and v is a function of x.

Example 26 : Find $f'(x)$ if $f(x) = sin(tan\ x)$

Solution : We have $g(t) = sin\ t$ and $t = h(x) = tanx$

$\therefore\ f(x) = (goh)(x) = g(h(x)) = sin(tanx)$

$\therefore\ f'(x) = g'(h(x))\ h'(x)$

$= g'(t)\ h'(x)$

$= cost\ h'(x)$

$= cos(tanx)\ sec^2x$ **($t = tanx$)**

$\therefore\ f'(x) = cos(tanx)\ sec^2x$

But we can make it simpler.

$f(u) = sinu$ where $u = tanx$

$\therefore\ f'(x) = \frac{df}{du} \frac{du}{dx} = cosu\ sec^2x = cos(tanx)\ sec^2x$

Generally, we make calculations orally.

Go on differentiating functions selecting the outermost function first and then proceeding to differentiate till we reach the variable and multiply all derivatives.

Let $f(x) = sin(cos(2x + 3))$

$\therefore\ f'(x) = cos\ (cos(2x + 3))\quad (-sin(2x + 3))\quad \cdot\quad 2$

Derivative of outer most function at its variable. (Proceed to 'inside') (Derivative of last function 2x + 3)

$= -2sin\ (2x + 3)\ cos(cos(2x + 3))$ **(rearrange)**

Let $f(x) = sin(tan\ (cos\ (x^2 - 3x + 51)))$

$\therefore\ f'(x) = cos(tan\ (cos\ (x^2 - 3x + 51)))\ (sec^2\ (cos\ (x^2 - 3x + 51)))\ (-sin\ (x^2 - 3x + 51)) \times (2x - 3)$

Stage 1 Stage 2 Stage 3 Stage 4

$= -(2x + 3)\ sin(x^2 - 3x + 51)\ sec^2\ (cos\ (x^2 - 3x + 51))\ cos(tan(cos(x^2 - 3x + 51)))$

(rearranging)

Example 27 : Find $\frac{dy}{dx}$, if $y = sin^3x\ cos^5x$

Solution : $\frac{dy}{dx} = sin^3x \frac{d}{dx} cos^5x + cos^5x \frac{d}{dx} sin^3x$

$= sin^3x \frac{d}{dx} (cosx)^5 + cos^5x \frac{d}{dx} (sinx)^3$

$= sin^3x \cdot 5cos^4x\ (-sinx) + cos^5x \cdot 3sin^2x\ cosx$

$= -5sin^4x \cdot cos^4x + 3sin^2x\ cos^6x$

[**Note :** In sin^nx, $sin^nx = (sinx)^n$; power is 'outermost' function.]

Example 28 : Find $\frac{d}{dx} sin^3(x^2 - x + 1)$

Solution : $\frac{d}{dx} sin^3(x^2 - x + 1) = \frac{d}{dx} [sin(x^2 - x + 1)]^3$

$= 3sin^2(x^2 - x + 1)\ cos(x^2 - x + 1)\ (2x - 1)$

$= 3(2x - 1)\ sin^2(x^2 - x + 1)\ cos(x^2 - x + 1)$

Example 29 : Find $\frac{d}{dx} \sqrt{sinx^3}$

Solution : $\frac{d}{dx} \sqrt{sinx^3} = \frac{d}{dx} (sinx^3)^{\frac{1}{2}}$

$= \frac{1}{2}(sinx^3)^{-\frac{1}{2}} \cdot cosx^3 \cdot 3x^2$ **($\sqrt{\ }$ is outermost function)**

$= \frac{3}{2} \frac{x^2\ cosx^3}{\sqrt{sinx^3}}$

(**Note :** Remember $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$)

Example 30 : Find $\frac{d}{dx} \sqrt[4]{sin^3x}$

Solution : $\frac{d}{dx} \sqrt[4]{sin^3x} = \frac{d}{dx} [(sinx)^3]^{\frac{1}{4}} = \frac{d}{dx} (sinx)^{\frac{3}{4}}$

$= \frac{3}{4} sin^{-\frac{1}{4}}x \cdot cosx$

$= \frac{3cosx}{4\sqrt[4]{sinx}}$

Exercise 5.3

Find the derivative of the following functions defined on proper domains :

1. $sin^3(2x + 3)$
2. tan^3x
3. $sin^3x\ cos^5x$
4. $cos(sin(sec(2x + 3)))$
5. $sec(cot(x^3 - x + 2))$
6. Differentiate the identity $sin3x = 3sinx - 4sin^3x$. What do you observe ?

7. Find $\frac{d}{dx}(2x+3)^m(3x+2)^n$ **8.** Find $\frac{d}{dx}(\sin^n x - \cos^n x)$

9. Find $\frac{d}{dx}\sin^3 x\cos^3 x$

10. Find $\frac{d}{dx}\sin^3(4x-1)\cos^3(2x+3)$

*

5.7 Derivative of Inverse Functions

We have studied inverse trigonometric functions in chapter 2. Now we would like to find their derivatives.

Derivative of Inverse Function : Let $f : (a, b) \to (c, d)$ be a one-one and onto function, so that its inverse function exists. Its inverse

$g : (c, d) \to (a, b)$ satisfies $x = g(y) = f^{-1}(y)$ if $y = f(x)$

We assume $f'(x) = \frac{dy}{dx} = \frac{1}{g'(y)} = \frac{1}{\frac{dx}{dy}}$ $\left(\frac{dx}{dy} \neq 0\right)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ **or** $f'(x) = \frac{1}{\frac{d}{dy}f^{-1}(y)}$

We have some standard forms :

(1) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ $\quad |x| < 1$

Let $y = \sin^{-1}x$. $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So $x = \sin y$ $\quad$ $\left(y \neq \pm\frac{\pi}{2} \text{ as } x \neq \pm 1\right)$

$\frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y}$ $\quad$ $\left(\cos y > 0 \text{ as } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$= \sqrt{1-x^2}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$ $\quad |x| < 1$

Let $y = \cos^{-1}x$. $y \in (0, \pi)$. So $x = \cos y$ $\quad$ $(y \neq 0, \pi \text{ as } x \neq \pm 1)$

$\frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y}$ $\quad$ $(\sin y > 0 \text{ as } y \in (0, \pi))$

$= -\sqrt{1-x^2}$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{d}{dx} \cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

or

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\therefore \quad \frac{d}{dx} \sin^{-1}x + \frac{d}{dx} \cos^{-1}x = \frac{d}{dx} \frac{\pi}{2} = 0$

$\therefore \quad \frac{d}{dx} \cos^{-1}x = -\frac{d}{dx} \sin^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad |x| < 1$

(3) $\frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2} \qquad x \in \mathbf{R}$

Let $y = \tan^{-1}x$. $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So $x = \tan y$.

$\therefore \quad \frac{dx}{dy} = \sec^2 y$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$\therefore \quad \frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2}$

(4) $\frac{d}{dx} \cot^{-1}x = -\frac{1}{1+x^2} \qquad x \in \mathbf{R}$

We can prove as in (3) or $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ will give the result.

(5) $\frac{d}{dx} \sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$

Let $y = \sec^{-1}x$. $y \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$. So, $x = \sec y$. **(Why $y \neq 0, y \neq \pi$?)**

$\therefore \quad \frac{dx}{dy} = \sec y \tan y$

Now, $\sec y = x, \quad y \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$

There are two cases. $y \in \left(0, \frac{\pi}{2}\right)$ or $y \in \left(\frac{\pi}{2}, \pi\right)$.

(i) $y \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad x = \sec y > 0, \ \tan y = \sqrt{x^2-1}$ as $\tan y > 0$

$\therefore \quad \frac{dx}{dy} = \sec y \tan y = x\sqrt{x^2-1} = |x|\sqrt{x^2-1}$. Since $x > 0$, so $|x| = x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

(ii) $y \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore \quad x = \sec y < 0$. So $|x| = -x$

$tany = -\sqrt{x^2 - 1}$, since $tany < 0$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{secy\ tany} = \frac{1}{-x\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}} \qquad \forall x \text{ such that } |x| > 1$$

(6) Similarly we can prove, $\frac{d}{dx} cosec^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$ $\qquad |x| > 1$

or since $sec^{-1}x + cosec^{-1}x = \frac{\pi}{2}$

$$\frac{d}{dx} sec^{-1}x + \frac{d}{dx} cosec^{-1}x = \frac{d}{dx}\frac{\pi}{2} = 0$$

$$\frac{d}{dx} cosec^{-1}x = -\frac{d}{dx} sec^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}} \qquad |x| > 1$$

We have introduced e in this chapter. $2 < e < 3$, e is the base of natural logarithm.

We assume $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$

We know $\lim\limits_{x \to 0} \frac{\log(1+x)}{x} = 1$ **(i)**

Let $\log_e(1 + x) = h$. So $x = e^h - 1$.

$\therefore$ Using (i), $\lim\limits_{h \to 0} \frac{h}{e^h - 1} = 1$ **(As $x \to 0$, $h = \log(1 + x) \to 0$)**

$$\therefore \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(7) $\frac{d}{dx} e^x = e^x$

$$\therefore \quad \frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = e^x \cdot 1 = e^x$$

$$\therefore \quad \frac{d}{dx} e^x = e^x$$

(8) $\frac{d}{dx} a^x = a^x \log_e a$

We know $a = e^{\log_e a}$

$\therefore \quad a^x = (e^{\log_e a})^x = e^{x \log_e a}$

$a^x = e^t$. Here $t = x \log_e a$

By chain rule $\frac{d}{dx} a^x = \frac{d}{dt} e^t \cdot \frac{dt}{dx}$

$= e^t \cdot \log_e a$ $\qquad (\frac{d}{dx} kx = k)$

$= a^x \log_e a$

$$\frac{d}{dx} a^x = a^x \log_e a$$

Note : By using chain rule $\frac{d}{dx} e^{\sin x} = e^{\sin x} \cos x$.

It is like this $e^{\sin x} = \exp(\sin x)$ **(exponential ($\sin x$))**

Outermost function is exp. $\frac{d}{dx}(\exp x) = \frac{d}{dx} e^x = e^x = \exp x$

$\therefore \quad \frac{d}{dx} e^{\sin x} = \frac{d}{dx} \exp(\sin x) = \exp(\sin x) \frac{d}{dx} \sin x = e^{\sin x} \cos x$

$\therefore \quad \frac{d}{dx} e^{\tan 2x} = e^{\tan 2x} \frac{d}{dx} \tan 2x$

$= 2e^{\tan 2x} \sec^2 2x$

(9) $\frac{d}{dx} \log_e x = \frac{1}{x}$ $\qquad x \in \mathbb{R}^+$

Let, $y = \log_e x$

$\therefore \quad x = e^y$

$\therefore \quad \frac{dx}{dy} = e^y$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$

$\therefore \quad \frac{d}{dx} \log_e x = \frac{1}{x}$

Example 31 : Find $\frac{d}{dx} \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$ $\qquad |x| < \frac{1}{\sqrt{3}}$

Solution : Let $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$|x| < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

$\Rightarrow \tan\left(-\frac{\pi}{6}\right) < \tan\theta < \tan\frac{\pi}{6}$

$\Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6}$ $\qquad$ (since $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan$ is $\uparrow$)

$\Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$

Now, $y = \tan^{-1} \frac{3x - x^3}{1 - 3x^2} = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$= 3\theta$ $\qquad \left(3\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$= 3\tan^{-1} x$

$\therefore \quad \frac{dy}{dx} = \frac{3}{1 + x^2}$

Example 32 : Find $\frac{d}{dx} sin^{-1} 2x\sqrt{1-x^2}$, $\quad |x| < \frac{1}{\sqrt{2}}$

Solution : Let $\theta = sin^{-1}x$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So $x = sin\theta$.

$|x| < \frac{1}{\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Now, $sin\left(-\frac{\pi}{4}\right) < sin\theta < sin\frac{\pi}{4}$

$\therefore \quad -\frac{\pi}{4} < \theta < \frac{\pi}{4}$ $\qquad\qquad \left(sin \text{ is } \uparrow \text{ in } \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right)$

$\therefore \quad -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

$\therefore \quad y = sin^{-1} 2x\sqrt{1-x^2}$

$= sin^{-1}(2sin\theta \; cos\theta)$ $\qquad \left(\sqrt{1-x^2} = \sqrt{1-sin^2\theta} = cos\theta \text{ as } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$= sin^{-1}(sin2\theta)$

$= 2\theta$ $\qquad \left(2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$\therefore \quad y = 2sin^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

Example 33 : Find $\frac{d}{dx} sec^{-1} \frac{1}{2x^2-1}$, $\quad 0 < x < \frac{1}{\sqrt{2}}$

Solution : Let $\theta = cos^{-1}x$. $\theta \in (0, \pi)$. So $x = cos\theta$. $\qquad$ (Why $\theta \ne 0$ or π ?)

$\therefore \quad y = sec^{-1}\frac{1}{2x^2-1} = sec^{-1}\frac{1}{2cos^2\theta - 1} = sec^{-1}\frac{1}{cos\,2\theta}$

$\therefore \quad y = sec^{-1}(sec2\theta)$

Now, $0 < x < \frac{1}{\sqrt{2}} \Rightarrow cos\frac{\pi}{2} < cos\theta < cos\frac{\pi}{4}$

$\Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2}$ $\qquad (cos \text{ is } \downarrow)$

$\Rightarrow \frac{\pi}{2} < 2\theta < \pi$

$\therefore \quad y = sec^{-1}(sec2\theta) = 2\theta = 2cos^{-1}x$ $\qquad \left(2\theta \in \left(\frac{\pi}{2}, \pi\right) \subset [0, \pi] - \left\{\frac{\pi}{2}\right\}\right)$

$\therefore \quad \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$

Example 34 : Find $\frac{d}{dx} cos^{-1}(4x^3 - 3x)$ for (i) $\frac{1}{2} < x < 1$ (ii) $0 < x < \frac{1}{2}$

Solution : Let $\theta = cos^{-1}x$ so that $x = cos\theta$, $\quad 0 < \theta < \pi$

$\therefore \quad y = cos^{-1}(4x^3 - 3x) = cos^{-1}(4cos^3\theta - 3cos\theta)$

$y = cos^{-1}(cos3\theta)$

(i) $\frac{1}{2} < x < 1 \Rightarrow cos\frac{\pi}{3} < cos\theta < cos0$

$\Rightarrow 0 < \theta < \frac{\pi}{3}$ (*cos* is ↓)

$\Rightarrow 0 < 3\theta < \pi$

$\therefore \quad y = cos^{-1}(cos3\theta) = 3\theta = 3cos^{-1}x$ $(3\theta \in (0, \pi))$

$\therefore \quad \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$

(ii) $0 < x < \frac{1}{2} \Rightarrow cos\frac{\pi}{2} < cos\theta < cos\frac{\pi}{3}$

$\Rightarrow \frac{\pi}{3} < \theta < \frac{\pi}{2}$ (*cos* is ↓)

$\Rightarrow \pi < 3\theta < \frac{3\pi}{2}$

$\Rightarrow 0 < 3\theta - \pi < \frac{\pi}{2}$

$\therefore \quad y = cos^{-1}(cos3\theta) = cos^{-1}(-cos(\pi - 3\theta))$

$= \pi - cos^{-1}(cos(\pi - 3\theta))$

$= \pi - cos^{-1}(cos(3\theta - \pi))$

$= \pi - (3\theta - \pi)$ $\left((3\theta - \pi) \in \left(0, \frac{\pi}{2}\right) \subset [0, \pi]\right)$

$= 2\pi - 3\theta$

$= 2\pi - 3cos^{-1}x$

$\therefore \quad \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

5.8 Derivative of an Implicit Function

Sometimes we encounter equations of type $f(x, y) = 0$ from which we may or may not get y as a function of x. Functions of type $y = sin^2x$ are called explicit functions of x. But $3y - sin2x = 0$ gives $y = \frac{1}{3}sin2x$.

This is an example of y being an implicit function of x.

Consider the circle $x^2 + y^2 = 1$.

It is not a graph of a function. But $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ two implicit functions can be defined from the relation $x^2 + y^2 - 1 = 0$.

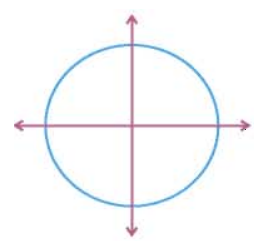

Figure 5.24

So we get two implicit functions. See that any vertical line meets the circle in two points but meets the semicircles in each semiplane of X-axis in only one point. So, each semicircle is a graph of an implicit function.

But some equations are not easy to solve.

$x^3 + y^3 = 3axy$ is such a relation. How to find $\frac{dy}{dx}$ for such implicit functions y ? We use the chain rule and differentiate the relation assuming that y is an implicit function of x.

For example $\frac{d}{dx}x^4 = 4x^3$

$$\frac{d}{dx}y^4 = \frac{d}{dy}y^4\frac{dy}{dx} = 4y^3\frac{dy}{dx}$$

So, when we differentiate a term involving variable y *w.r.t* x, we follow usual rules of differentiation and multiply the result by $\frac{dy}{dx}$.

Let us solve some examples.

Example 35 : Find $\frac{dy}{dx}$ from $x + y = sinxy$

Solution : Differentiating the equation,

$$\frac{d}{dx}x + \frac{d}{dx}y = \frac{d}{dx}sinxy$$

$\therefore \quad 1 + \frac{dy}{dx} = cosxy\frac{d}{dx}(xy)$ **(chain rule)**

$= cosxy\left(x\frac{d}{dx}y + y \cdot 1\right)$ **(product rule)**

$$\therefore \quad 1 + \frac{dy}{dx} = x\,cosxy\frac{dy}{dx} + ycosxy$$

$$\therefore \quad (1 - xcos\,xy)\frac{dy}{dx} = y\,cosxy - 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{ycosxy - 1}{1 - xcosxy}$$

Example 36 : Find $\frac{dy}{dx}$ for $x^3 + y^3 = 3axy$

Solution : $3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx} + y \cdot 1\right)$

$$\therefore \quad (y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Example 37 : Find $\frac{dy}{dx}$ from $ax^2 + 2hxy + by^2 = 100$

Solution : $2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0$

$$\therefore \quad (hx + by)\frac{dy}{dx} = -(ax + hy)$$

$$\therefore \quad \frac{dy}{dx} = -\left(\frac{ax + hy}{hx + by}\right)$$

Example 38 : Find $\frac{dy}{dx}$ from $sin^2x + sin^2y = 1$.

Solution : $2sinx\ cosx + 2siny\ cosy\frac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{-\sin 2x}{\sin 2y}$$

or

$$\sin^2 y = 1 - \sin^2 x = \cos^2 x$$

$$\therefore \quad \sin y = \pm \cos x \qquad \text{(Two functions)}$$

$$\therefore \quad \cos y \frac{dy}{dx} = \mp \sin x$$

$$\therefore \quad \frac{dy}{dx} = \pm \frac{\sin x}{\cos y}$$

Note : If $\sin^2 x + \sin^2 y = 2$, then $\sin^2 x = \sin^2 y = 1$ as $|\sin x| \le 1$, $|\sin y| \le 1$. No such function exists. If $\sin^2 x + \sin^2 y = 3$, then can we write $\frac{dy}{dx} = \frac{-\sin 2x}{\sin 2y}$?

No. $\sin^2 x + \sin^2 y < 2$. No implicit function exists if $\sin^2 x + \sin^2 y = 3$. We assume existence of implicit function and differentiate. But an implicit function may not exist.

Exercise 5.4

Find $\frac{dy}{dx}$: (1 to 10)

1. $x^2 + y^2 = 1$
2. $x + \sin x = \sin y$
3. $\sin(x + y) = x - y$
4. $2x^2 + 3xy + y^2 = 1$
5. $\sin x + \sin y = \tan xy$
6. $\frac{x^2}{4} - \frac{y^2}{9} = 1$
7. $y^2 = 10x$
8. $\frac{x^2}{16} + \frac{y^2}{25} = 1$
9. $x^2 + y^2 - 4x - 6y - 25 = 0$
10. $\sin x = \sin y$

Find the derivative : (11 to 16)

11. $y = \sin^{-1}(3x - 4x^3)$, $\quad 0 < x < \frac{1}{2}$
12. $y = \tan^{-1}\frac{2x}{1-x^2}$, $\quad x \ne \pm 1$
13. $y = \cos^{-1}\frac{1-x^2}{1+x^2}$
14. $y = \sin^{-1}\frac{2x}{1+x^2}$
15. $y = \tan^{-1}\frac{3x-x^3}{1-3x^2}$, $\quad |x| > \frac{1}{\sqrt{3}}$
16. $y = \sin^{-1} 2x\sqrt{1-x^2}$, $\quad \frac{1}{\sqrt{2}} < x < 1$

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5.9 Parametric Differentiation

Sometimes x and y are given as functions of another variable, say t, called a parameter.

Let $x = f(t) \quad y = g(t)$

Assuming that we can obtain $t = f^{-1}(x)$ and substituting in $y = g(t)$, we get $y = g(f^{-1}(x))$.

So, y is a function of x.

But this type of solving and differentiating would be cumbersome. We have the following rule :

Rule for differentiation of parametric functions :

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)} \quad \text{where } f'(t) \neq 0$$

Example 39 : If $x = acos\theta$, $y = bsin\theta$, find $\frac{dy}{dx}$.

Solution : $\frac{dx}{d\theta} = -asin\theta$, $\frac{dy}{d\theta} = bcos\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{bcos\,\theta}{asin\,\theta} = \frac{-b}{a}cot\theta$$

$$\frac{dy}{dx} = -\frac{bcos\,\theta}{asin\,\theta} = \frac{-b}{a}\left(\frac{\frac{x}{a}}{\frac{y}{b}}\right) = -\frac{b^2x}{a^2y}$$

or directly $\frac{x^2}{a^2} + \frac{y^2}{b^2} = cos^2\theta + sin^2\theta = 1$

$$\therefore \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Example 40 : If $x = at^2$, $y = 2at$, find $\frac{dy}{dx}$.

Solution : $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t} \quad (t \neq 0)$$

Example 41 : If $x = asin^3\theta$, $y = bcos^3\theta$, find $\frac{dy}{dx}$.

Solution : $\frac{dx}{d\theta} = 3asin^2\theta\ cos\theta$, $\frac{dy}{d\theta} = 3bcos^2\theta\,(-sin\theta)$

$$\therefore \quad \frac{dy}{dx} = \frac{-3bcos^2\theta\,sin\,\theta}{3asin^2\theta\,cos\,\theta} = \frac{-b}{a}cot\theta$$

$cot^3\theta = \frac{cos^3\theta}{sin^3\theta} = \frac{ay}{bx}$. So $cot\theta = \left(\frac{ay}{bx}\right)^{\frac{1}{3}}$.

So $\frac{dy}{dx} = \frac{-b}{a}\left(\frac{ay}{bx}\right)^{\frac{1}{3}}$

$$= -\frac{b^{\frac{2}{3}}\,y^{\frac{1}{3}}}{a^{\frac{2}{3}}\,x^{\frac{1}{3}}}$$

or $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = cos^2\theta + sin^2\theta = 1$

$\therefore \quad \frac{2}{3}\frac{x^{-\frac{1}{3}}}{a^{\frac{2}{3}}} + \frac{2}{3}\frac{y^{-\frac{1}{3}}}{b^{\frac{2}{3}}}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{b^{\frac{2}{3}}\, y^{\frac{1}{3}}}{a^{\frac{2}{3}}\, x^{\frac{1}{3}}}$

Exercise 5.5

Find $\frac{dy}{dx}$: (wherever y is defined as a function of x and $\frac{dx}{dt}$ or $\frac{dx}{d\theta} \neq 0$)

1. $x = a\sec\theta,\ y = b\tan\theta \qquad \theta \in R - \left[\left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\} \cup \{k\pi \mid k \in Z\}\right]$
2. $x = \cos\theta - \cos2\theta \qquad y = \sin\theta - \sin2\theta \qquad \theta \in R - \{k\pi \mid k \in Z\},\ \cos\theta \neq \frac{1}{4}$
3. $x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta)$
4. $x = a(\cos t + \log \tan\frac{t}{2}), \qquad y = a\sin t$
5. $x = a(\cos\theta + \theta\sin\theta), \qquad y = a(\sin\theta - \theta\cos\theta)$
6. $x = \frac{a}{t^2} \qquad y = bt$
7. If $x = \sqrt{a^{\sin^{-1}t}}, \qquad y = \sqrt{a^{\cos^{-1}t}}$, prove $\frac{dy}{dx} = \frac{-y}{x} \qquad |t| < 1$

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5.10 Logarithmic Differentiation

Sometimes we have to differentiate a product of several functions or a complicated product or $[f(x)]^{g(x)}$ form.

In such a case, it is customary to find $\frac{dy}{dx}$ by taking logarithms.

Example 42 : Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{(2x+3)(3x-4)}{(4x+9)(x-8)}}$

Solution : $\log y = \frac{1}{2}\,[\log(2x+3) + \log(3x-4) - \log(4x+9) - \log(x-8)]$

$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{2}{2x+3} + \frac{3}{3x-4} - \frac{4}{4x+9} - \frac{1}{x-8}\right]$

$\therefore \quad \frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{2x+3} + \frac{3}{3x-4} - \frac{4}{4x+9} - \frac{1}{x-8}\right]$

Example 43 : Find $\frac{dy}{dx}$ if $y = x^{\sin x}$

Solution : $\log y = \sin x \log x$

$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \cos x \log x$

$\therefore \quad \frac{dy}{dx} = \left[\frac{\sin x}{x} + \cos x \log x\right] y$

Example 44 : If $x^y + y^x + a^x + x^a = 1$, find $\frac{dy}{dx}$.

Solution : Let $u = x^y$, $v = y^x$, $w = a^x + x^a$

Now, $\log u = y \log x$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x \frac{dy}{dx}$$

$$\therefore \quad \frac{du}{dx} = \left(\frac{y}{x} + \log x \frac{dy}{dx}\right) x^y$$

Now, $v = y^x$

$$\therefore \quad \log v = x \log y$$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\therefore \quad \frac{dv}{dx} = \left(\frac{x}{y}\frac{dy}{dx} + \log y\right) y^x$$

Now, $u + v + w = 1$

$$\therefore \quad \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$$

$$\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) x^y + \left(\frac{x}{y}\frac{dy}{dx} + \log y\right) y^x + a^x \log_e a + ax^{a-1} = 0$$

$$\left(x^y \log x + \frac{x}{y} y^x\right) \frac{dy}{dx} = -\left(\frac{x^y \cdot y}{x} + y^x \log y + a^x \log a + ax^{a-1}\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(yx^{y-1} + y^x \log y + a^x \log a + ax^{a-1})}{xy^{x-1} + x^y \log x}$$

Example 45 : Find $\frac{dy}{dx}$ if $y = (sinx)^x + sinx^x$

Solution : Let $u = (sinx)^x = e^{x \log sinx}$

(since $a = e^{\log_e a}$, $sinx = e^{\log sinx}$)

$$\therefore \quad \frac{du}{dx} = e^{x \log sinx} \frac{d}{dx}(x \log sinx)$$

$$= e^{x \log sinx}\left(1 \cdot \log sinx + \frac{xcosx}{sinx}\right)$$

$$= (sinx)^x (\log sinx + xcot\, x)$$

$$\therefore \quad \frac{d}{dx} sinx^x = cosx^x \frac{d}{dx} x^x$$

$$= cosx^x \frac{d}{dx} e^{x \log x}$$

$$= cosx^x \cdot e^{x \log x}\left(x \frac{1}{x} + \log x\right)$$

$$= x^x cosx^x (1 + \log x)$$

$$\therefore \quad \frac{dy}{dx} = (sinx)^x (\log sinx + xcot\, x) + x^x cosx^x (1 + \log x)$$

(**Note :** $a = e^{\log_e a}$ helps to avoid taking logarithms.)

Exercise 5.6

Find $\frac{dy}{dx}$:

1. $y = \left(x+\frac{1}{x}\right)^x + \left(x+\frac{1}{x}\right)^{\frac{1}{x}}$

2. $y = cosx^x + sinx^x$

3. $y = \sqrt[3]{\frac{(2x+1)^3(4x+3)^5}{(7x-1)^6}}$

4. $y = (\log x)^{cosx}$

5. $y = (x+1)^2(x+2)^3(x+3)^4$

6. $y = (\log x)^x + \log x^x$

7. $y = x^{x \sin x} + (\sin x)^x$

8. $y = x^{\left(x+\frac{1}{x}\right)}$

9. $y = (\sin x)^x + \left(\frac{1}{x}\right)^{cosx}$

10. $y = 3^{sinx} + 4^{cosx}$

11. $y^x = x^y$

12. $xy = e^{x-y}$

13. $x^y y^x = 1$

14. $y = (1+x)(1+x^2)(1+x^4)(1+x^8)$

15. If $y = (x^2 - 2x + 3)(x^2 - 3x + 15)$, find $\frac{dy}{dx}$

by (1) Product rule

(2) Multiply and using rule for polynomials.

(3) Logarithmic differentiation

and compare.

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5.11 Second Order Derivative

If f is a differentiable function of x on (a, b) and if $f'(x)$ is also a differentiable function of x on (a, b), its derivative is called second derivative of f and is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$ or y_2 where $y = f(x)$.

Thus $f''(x) = \frac{d}{dx}f'(x)$ or $\frac{d^2y}{dx^2}$, or y_2. Here y_1 denotes $f'(x)$ or $\frac{dy}{dx}$.

We can use chain rule as follows :

$$\frac{d}{dx}y^2 = \frac{d}{dy}y^2\frac{dy}{dx} = 2y\frac{dy}{dx} = 2yy_1$$

$$\frac{d}{dx}{y_1}^2 = \frac{d}{dy_1}{y_1}^2\frac{d}{dx}y_1 = 2y_1\frac{dy_1}{dx} = 2y_1y_2$$

Remember $\frac{d}{dx}y^2 = 2yy_1$, $\frac{d}{dx}{y_1}^2 = 2y_1y_2$

Example 46 : If $y = acosx + bsinx$, prove $\frac{d^2y}{dx^2} + y = 0$

Solution : $y = acosx + bsinx$

$\therefore \quad y_1 = -a\sin x + b\cos x$

$\therefore \quad y_2 = -a\cos x - b\sin x = -y$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

Example 47 : $y = ae^{4x} + be^{5x}$, prove $y_2 - 9y_1 + 20y = 0$

Solution : $y = ae^{4x} + be^{5x}$

$y_1 = 4ae^{4x} + 5be^{5x}$

$\therefore \quad y_2 = 16ae^{4x} + 25be^{5x}$

$$\therefore \quad y_2 - 9y_1 + 20y = (16ae^{4x} + 25be^{5x}) - 9(4ae^{4x} + 5be^{5x}) + 20(ae^{4x} + be^{5x})$$

$$= (16 - 36 + 20)\, ae^{4x} + (25 - 45 + 20)\, be^{5x} = 0$$

$\therefore \quad y_2 - 9y_1 + 20y = 0$

Example 48 : $y = x^4 + \sin^3 x$. Find $\frac{d^2y}{dx^2}$.

Solution : $y = x^4 + \sin^3 x$

$$\frac{dy}{dx} = 4x^3 + 3\sin^2 x \cos x$$

$$\therefore \quad \frac{d^2y}{dx^2} = 12x^2 + 6\sin x \cos^2 x + 3\sin^2 x (-\sin x)$$

$$= 12x^2 + 6\sin x \cos^2 x - 3\sin^3 x$$

Example 49 : Find $\frac{d^2y}{dx^2}$ for $y = \log(\log x)$.

Solution : $y = \log(\log x)$

$$\frac{d}{dx}\log(\log x) = \frac{1}{\log x}\,\frac{1}{x} = \frac{1}{x\log x}$$

$$\therefore \quad \frac{d^2}{dx^2}\log(\log x) = \frac{(x\log x)\,0 - 1\cdot(1\log x + x\cdot\frac{1}{x})}{(x\log x)^2}$$

$$= \frac{-(1 + \log x)}{(x\log x)^2}$$

Example 50 : If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$.

Solution : $y = a\cos(\log x) + b\sin(\log x)$

$$y_1 = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$\therefore \quad xy_1 = -a\sin(\log x) + b\cos(\log x)$

$$\therefore \quad \frac{d}{dx}(xy_1) = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$\therefore \quad x(xy_2 + 1\cdot y_1) = -a\cos(\log x) - b\sin(\log x) = -y$

$\therefore \quad x^2y_2 + xy_1 + y = 0$

Example 51 : If $y = cos^{-1}x$, prove $(1 - x^2)y_2 - xy_1 = 0$.

Solution : $y = cos^{-1}x$

$\therefore \quad y_1 = \frac{-1}{\sqrt{1-x^2}}$

$\therefore \quad (1 - x^2){y_1}^2 = 1$

$\therefore \quad \frac{d}{dx}(1 - x^2){y_1}^2 = 0$

$\therefore \quad (1 - x^2)2y_1y_2 + (-2x{y_1}^2) = 0$

$\therefore \quad (1 - x^2)y_2 - xy_1 = 0$ $\quad (y_1 \neq 0)$

Example 52 : If $y = tan^{-1}x$, prove $(1 + x^2)y_2 + 2xy_1 = 0$.

Solution : $y = tan^{-1}x$

$\therefore \quad y_1 = \frac{1}{1+x^2}$

$\therefore \quad (1 + x^2)y_1 = 1$

$\therefore \quad (1 + x^2)y_2 + 2xy_1 = 0$

Example 53 : If $y = ae^{px} + be^{qx}$, prove that $y_2 - (p + q)y_1 + pqy = 0$.

Solution : $y_1 = ape^{px} + bqe^{qx}$

$y_2 = ap^2e^{px} + bq^2e^{qx}$

$ape^{px} + bqe^{qx} - y_1 = 0$ $\quad$ (i)

$ap^2e^{px} + bq^2e^{qx} - y_2 = 0$ $\quad$ (ii)

Solving (i) and (ii) for e^{px} and e^{qx},

$$e^{px} = \frac{-bqy_2 + bq^2y_1}{abpq^2 - abp^2q}$$

$$\therefore \quad e^{px} = \frac{-y_2 + qy_1}{ap(q-p)}$$

$$e^{qx} = -\frac{-apy_2 + ap^2y_1}{abpq(q-p)}$$

$$e^{qx} = -\frac{-y_2 + py_1}{bq(q-p)}$$

$\therefore$ Substituting in $y = ae^{px} + be^{qx}$

$$y = \left(\frac{-y_2 + qy_1}{p(q-p)}\right) - \left(\frac{-y_2 + py_1}{q(q-p)}\right)$$

$$\therefore \quad pq(q - p)y = -qy_2 + q^2y_1 + py_2 - p^2y_1$$

$$= (p - q)y_2 - (p^2 - q^2)y_1$$

$$\therefore \quad y_2 - (p + q)y_1 + pqy = 0$$

5.12 Mean Value Theorems

There are some important theorems in differential calculus called mean value theorems.

Rolle's Theorem : If f is continuous in $[a, b]$ and differentiable in (a, b) and if $f(a) = f(b)$,

then there exists some $c \in (a, b)$ for which $f'(c) = 0$

Geometrical Interpretation : If the graph of $y = f(x)$ is continuous in $[a, b]$ and if it has a non-vertical tangent at all points $(x, f(x))$ where $x \in (a, b)$ and if $f(a) = f(b)$, there is some $c \in (a, b)$ such that tangent at $(c, f(c))$ to thecurve $y = f(x)$ is horizontal or we can say it is X-axis or parallel to X-axis.

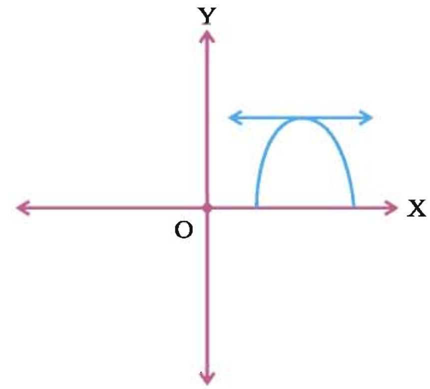

Figure 5.25

Mean-value Theorem (Lagrange) : If f is continuous in $[a, b]$ and differentiable in (a, b), then $\frac{f(b) - f(a)}{b - a} = f'(c)$ for some $c \in (a, b)$.

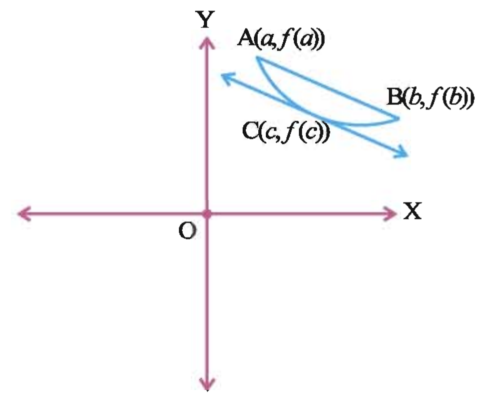

Figure 5.26

Geometric Interpretation : If the graph of $y = f(x)$ is continuous in $[a, b]$ and if $y = f(x)$ has a non-vertical tangent at all points, $(x, f(x))$ where $x \in (a, b)$, then $\exists c \in (a, b)$ such that tangent at $(c, f(c))$ is parallel to the secant line joining $A(a, f(a))$ and $B(b, f(b))$.

We know slope of $\overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$

Slope of tangent at $(c, f(c)) = f'(c)$.

Hence the result.

Example 54 : Verify Rolle's theorem for $f(x) = x^2 - 4x + 3$ in $[1, 3]$.

Solution : f is continuous in $[1, 3]$ and differentiable in $(1, 3)$ as it is a polynomial in x.

$f(1) = 0, f(3) = 9 - 12 + 3 = 0$

$\therefore \exists c \in (1, 3)$ such that $f'(c) = 0$

Now, $f'(c) = 2c - 4 = 0 \Rightarrow c = 2$ and $2 \in (1, 3)$

$\therefore c = 2, c \in (1, 3)$

Example 55 : Verify Rolle's theorem for $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$.

Solution : f is continuous in $[1, 3]$ and differentiable in $(1, 3)$ and $f(1) = 0 = f(3)$

$$f'(x) = 3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{12 \pm \sqrt{144 - 132}}{6}$$

$\therefore x = 2 \pm \frac{1}{\sqrt{3}} \in (1, 3)$

$\therefore$ The are two value of c namely $c = 2 \pm \frac{1}{\sqrt{3}}$. **$(c \in (1, 3))$**

Example 56 : Verify Rolle's theorem for $f(x) = sinx$ in $[0, \pi]$.

Solution : *sine* is continuous in $[0, \pi]$ and differentiable in $(0, \pi)$ and $sin\,0 = sin\pi = 0$

$$f'(x) = cosx = 0 \Rightarrow x = \frac{\pi}{2} \text{ in } [0, \pi].$$

$\therefore c = \frac{\pi}{2}$ and $\frac{\pi}{2} \in (0, \pi)$ **$(c \in (0, \pi))$**

Example 57 : Apply the mean value theorem to $f(x) = cosx$ over $[0, \pi]$.

Solution : *cos* is continuous in $[0, \pi]$ and differentiable in $(0, \pi)$

$a = 0,\ b = \pi$

$\frac{f(b) - f(a)}{b - a} = f'(c)$ gives, $\frac{cos\,\pi - cos\,0}{\pi - 0} = -sinc$

$\therefore\quad \frac{-1 - 1}{\pi} = -sinc$

$sinc = \frac{2}{\pi}$. Also $0 < \frac{2}{\pi} < 1$.

Since $\exists\, c,\ 0 < c < \pi$ such that $sinc = \frac{2}{\pi}$

[In fact, there will be two value of c in each of $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ such that $sinc = \frac{2}{\pi}$

If we take $c = sin^{-1}\frac{2}{\pi}$, we will get only one value of c in $\left(0, \frac{\pi}{2}\right)$.]

Example 58 : Apply the mean value theorem to $f(x) = e^x$ in $[0, 1]$.

Soloution : $f(x) = e^x$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$. $a = 0,\ b = 1$.

$\frac{f(b) - f(a)}{b - a} = f'(c)$ gives, $\frac{e - 1}{1 - 0} = e^c$

$\therefore\quad e^c = e - 1$

$\therefore\quad c = \log_e(e - 1)$

Now, $2 < e < 3$

$\therefore\quad 1 < e - 1 < 2$

$\therefore\quad 0 < \log(e - 1) < \log_e 2 < \log_e e = 1$ $\qquad (e > 2)$

$\therefore\quad c \in (0, 1)$ and $c = \log_e(e - 1)$

Example 59 : Apply the mean-value theorem to $f(x) = \log x$ in $[1, e]$.

Solution : log function is continuous in $[1, e]$ and differentiable in $(1, e)$.

$a = 1,\ b = e,\ f'(x) = \frac{1}{x}$

$\therefore\quad \frac{\log e - \log 1}{e - 1} = \frac{1}{c}$

$\therefore\quad \frac{1}{c} = \frac{1}{e - 1}$ $\qquad (\log 1 = 0,\ \log_e e = 1)$

$\therefore\quad c = e - 1$

Also $1 < e - 1 < e$ as $e > 2$

$\therefore\quad c = e - 1$ $\qquad (c \in (1, e))$

Example 60 : Can you apply the mean-value theorem and Rolle's theorem to $f(x) = [x]$ in $[-2, 2]$.

Solution : f is discontinuous at -1, 0, 1 and 2 (why not at -2 ?)

f is not differentiable at $-1, 0, 1$ in $(-2, 2)$.

$$f(x) = \begin{cases} -2 & -2 \le x < -1 \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & x = 2 \end{cases}$$

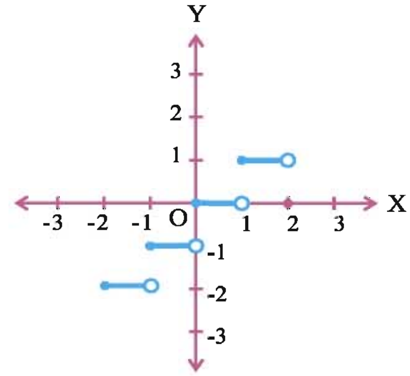

Figure 5.27

But $f'(x) = 0$, $x \in (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2)$

(Constant function)

$\therefore$ Conditions of Rolle's theorem are sufficient but not necessary.

Also $\frac{f(2) - f(-2)}{2-(-2)} = \frac{2-(-2)}{4} = 1 \neq f'(c)$ for any c in $(-2, 2)$.

(Infact either $f'(c)$ does not exist or $f'(c) = 0$ for $c \in (-2, 2)$.)

In any interval $[a, b]$ not containing an integer, f is a constant function and Rolle's theorem and mean-value theorem can be verified but not otherwise.)

Exercise 5.7

Verify Rolle's theorem : (1 to 8)

1. $f(x) = x(x-3)^2$ $\quad x \in [0, 3]$
2. $f(x) = x^3 - 6x^2 + 11x - 6$ $\quad x \in [1, 3]$
3. $f(x) = \sqrt{9 - x^2}$ $\quad x \in [-3, 3]$
4. $f(x) = \log\left(\frac{x^2 + ab}{x(a+b)}\right)$ $\quad x \in [a, b]$ $\quad 0 < a < b$
5. $f(x) = \sin x + \cos x - 1$ $\quad x \in \left[0, \frac{\pi}{2}\right]$
6. $f(x) = e^x(\sin x - \cos x)$ $\quad x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
7. $f(x) = a^{\sin x}$ $\quad x \in [0, \pi]$, $a > 0$
8. $f(x) = e^x \cos x$ $\quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Verify Mean Value Theorem : (9 - 10)

9. $f(x) = x - 2\sin x$, $\quad x \in [-\pi, \pi]$
10. $f(x) = \log_e x$, $\quad x \in [1, 2]$

11. Prove $\frac{x-y}{x} < \log_e \frac{x}{y} < \frac{x-y}{y}$, $0 < y < x$ using Mean Value theorem and taking $f(x) = \log_e x$.

12. Apply Mean Value theorem and find c :

(1) $f(x) = x + \frac{1}{x}$ $\quad x \in [1, 3]$

(2) $f(x) = tan^{-1}x$ $\quad x \in [0, 1]$

13. Prove $sec^2 a < \frac{tanb - tana}{b - a} < sec^2 b$ $\quad 0 < a < b < \frac{\pi}{2}$

14. Find a point on the graph of $y = (x - 4)^2$ where tangent is parallel to the line joining A(4, 0), B(5, 1).

*

Miscellaneous Example :

Example 61 : Find $\frac{d}{dx} \log_7 (\log_7 x)$.

Solution : $y = \log_7 \left(\frac{\log x}{\log 7}\right) = \log_7(\log x) - \log_7(\log 7)$

$\therefore \frac{dy}{dx} = \frac{d}{dx} \log_7 (\log x).$ $\quad (\frac{d}{dx} \log_7 (\log 7) = 0)$

$= \frac{d}{dx} \frac{\log(\log x)}{\log 7}$

$= \frac{1}{\log 7} \frac{d}{dx} \log(\log x)$

$= \frac{1}{\log 7} \frac{1}{\log x} \frac{1}{x} = \frac{1}{x \log x \log 7}$

Example 62 : Find $\frac{d}{dx} tan^{-1} \left(\frac{sinx}{1+cosx}\right)$ $\quad \pi < x < 2\pi$

Solution : $y = tan^{-1} \left(\frac{sinx}{1+cosx}\right)$

$= tan^{-1} \left(\frac{2 sin\frac{x}{2} cos\frac{x}{2}}{2cos^2\frac{x}{2}}\right)$

$= tan^{-1} \left(tan\frac{x}{2}\right)$ $\quad \frac{\pi}{2} < \frac{x}{2} < \pi$

Now, $\frac{\pi}{2} < \frac{x}{2} < \pi \Rightarrow -\frac{\pi}{2} < \frac{x}{2} - \pi < 0$

Now, $y = tan^{-1}\left(tan\left(\frac{x}{2}\right)\right) = tan^{-1}\left(tan\left(\frac{x}{2} - \pi\right)\right) = \frac{x}{2} - \pi$

$\therefore \frac{dy}{dx} = \frac{1}{2}$

Example 63 : If $f(x) = cos^{-1} \frac{1-9^x}{1+9^x}$, find $f'(x)$, $x \in$ R

Solution : Let $t = 3^x$

$$\therefore \quad f(t) = cos^{-1} \frac{1-t^2}{1+t^2}$$

Let $\theta = tan^{-1}t$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. So $t = tan\theta$

$3^x > 0$. So $t = tan\theta > 0$. So, $0 < \theta < \frac{\pi}{2}$

$$\therefore \quad 0 < 2\theta < \pi$$

$$\therefore \quad cos^{-1} \frac{1-t^2}{1+t^2} = cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)$$

$$= cos^{-1}(cos2\theta)$$

$$= 2\theta \qquad (0 < 2\theta < \pi)$$

$$= 2tan^{-1}t$$

$$\therefore \quad cos^{-1} \frac{1-9^x}{1+9^x} = 2tan^{-1}3^x \qquad (\text{Taking } t = 3^x)$$

$$\therefore \quad f(x) = cos^{-1} \frac{1-9^x}{1+9^x} = 2tan^{-1}3^x$$

$$\therefore \quad f'(x) = \frac{2 \cdot 3^x \log_e 3}{1+(3^x)^2} = \frac{2 \cdot 3^x \log_e 3}{1+3^{2x}}$$

Example 64 : If $x = a\,(cost + tsint)$, $y = a(sint - tcost)$, find $\frac{d^2y}{dx^2}$.

Solution : $\frac{dx}{dt} = a(-sint + tcost + sint) = at\ cost$

$$\frac{dy}{dt} = a(cost - cost + tsint) = at\ sint$$

$$\therefore \quad \frac{dy}{dx} = tant$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(tant)$$

$$= \frac{d}{dt}(tant)\ \frac{dt}{dx}$$

$$= \frac{sec^2t}{\frac{dx}{dt}}$$

$$= \frac{sec^2t}{a\,t\cos t} = \frac{sec^3t}{at}$$

Example 65 : If $y = e^{asin^{-1}x}$, $|\,x\,| \le 1$ prove that $(1 - x^2)y_2 - xy_1 - a^2y = 0$.

Solution : $\frac{dy}{dx} = y_1 = e^{asin^{-1}x}\ \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$

$\therefore \quad (1 - x^2){y_1}^2 = a^2y^2$

$\therefore \quad (1 - x^2)2y_1y_2 + (-2x){y_1}^2 = a^2 2yy_1$ $\quad \left(\frac{d}{dx} y^2 = 2yy_1, \frac{d}{dx} {y_1}^2 = 2y_1y_2 \text{ etc.}\right)$

$\therefore \quad (1 - x^2)y_2 - xy_1 - a^2y = 0$ $\quad (y_1 \neq 0)$

Example 66 : Does there exists a function continuous everywhere but not differentiable at exactly n real numbers ?

Solution : Let $f(x) = |x - 1| + |x - 2| + |x - 3| + ... + |x - n|$

$\therefore$ $|x|$ is continuous on R. So $|x - 1|$, $|x - 2|$,..., $|x - n|$ all are continuous on R, because composite function of continuous functions is continuous.

So, $f(x)$ is continuous on R, because it is a sum of continuous functions.

$|x - 1|$, $|x - 2|$,..., $|x - n|$ are differentiable except at $x = 1$, $x = 2$,..., $x = n$ respectively.

$|x - 2|$, $|x - 3|$,..., $|x - n|$ are differentiable at $x = 1$.

$\therefore$ $g(x) = |x - 2| + |x - 3| + ... + |x - n|$ is differentiable at $x = 1$.

If $f(x) = |x - 1| + |x - 2| + ... + |x - n|$ is differentiable at $x = 1$, then

$f(x) - g(x) = |x - 1|$ is differentiable at $x = 1$.

But $|x - 1|$ is not differentiable at $x = 1$.

$\therefore$ $f(x) = |x - 1| + |x - 2| + ... + |x - n|$ is not differentiable at $x = 1$.

Similarly $|x - 1| + |x - 2| + ... + |x - n|$ is not differentiable at $x = 2, 3, ..., n$.

$\therefore$ f is continuous on R but not differentiable at $x = 1, 2, 3, ..., n$.

Example 67 : $\sin y = x\sin(a + y)$. Prove $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$

Solution : $\cos y \frac{dy}{dx} = \sin(a + y) + x\cos(a + y) \frac{dy}{dx}$

$\therefore \quad [\cos y - x\cos(a + y]\frac{dy}{dx} = \sin(a + y)$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin(a + y)}{\cos y - x\cos(a + y)}$$

$$= \frac{\sin(a + y)}{\cos y - \frac{\sin y}{\sin(a + y)}\cos(a + y)}$$

$$= \frac{\sin^2(a + y)}{\sin(a + y)\cos y - \cos(a + y)\sin y}$$

$$= \frac{\sin^2(a+y)}{\sin a}$$

$(\sin(a + y) \cos y - \cos(a + y) \sin y = \sin(a + y - y) = \sin a)$

or

$$x = \frac{\sin y}{\sin(a+y)}$$

$$\therefore \quad \frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cos(a+y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Example 68 : If $(x - a)^2 + (y - b)^2 = r^2$, prove that $\left|\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\right|$ is a constant.

Solution : $2(x - a) + 2(y - b)y_1 = 0$

$$\therefore \quad y_1 = -\frac{x-a}{y-b}$$

$$\therefore \quad y_2 = -\frac{(y-b)\cdot 1 - (x-a)y_1}{(y-b)^2}$$

$$= -\frac{(y-b) + \frac{(x-a)(x-a)}{y-b}}{(y-b)^2}$$

$$= -\frac{(x-a)^2 + (y-b)^2}{(y-b)^3}$$

$$= -\frac{r^2}{(y-b)^3}$$

$$\therefore \quad \left|\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\right| = \left|\frac{\left[1 + \frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-r^2}{(y-b)^3}}\right|$$

$$= \left|\frac{[(x-a)^2 + (y-b)^2]^{\frac{3}{2}}}{-r^2}\right|$$

$$= \left|-\frac{r^3}{r^2}\right| = |r| \text{ is a constant.}$$

($\left|\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\right|$ is called the radius of curvature of curve $y = f(x)$ at any point $(x, f(x))$. Circle is a curve having 'uniform' radius of curvature at every point.)

Example 69 : Find $\frac{d}{dx}(\log x)^{\log x}$ wherever defined.

Solution : $y = (\log x)^{\log x}$

$\therefore \quad \log y = \log x\,(\log(\log x))$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\log(\log x) + \frac{\log x}{\log x}\,\frac{1}{x}$$

$$= \frac{\log(\log x) + 1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{1 + \log(\log x)}{x}\right)(\log x)^{\log x}$$

Example 70 : Find $\left[\frac{d}{dx} sec^{-1}x\right]_{x = -2}$ by definition. (First principle)

Solution : $\left[\frac{d}{dx} sec^{-1}x\right] = \lim\limits_{x \to -2} \frac{sec^{-1}x - sec^{-1}(-2)}{x - (-2)}$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - (\pi - sec^{-1}2)}{sect + 2} \qquad (t = sec^{-1}x)$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \left(\pi - \frac{\pi}{3}\right)}{sect + 2}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \frac{2\pi}{3}}{sect + 2}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \frac{2\pi}{3}}{2sect\left(cost - cos\frac{2\pi}{3}\right)}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{t - \frac{2\pi}{3}}{2sect\left(-2\,sin\frac{t + \frac{2\pi}{3}}{2}\,sin\frac{t - \frac{2\pi}{3}}{2}\right)}$$

$$= \lim_{t \to \frac{2\pi}{3}} \frac{\left(t - \frac{2\pi}{3}\right)/2}{-2sect \cdot sin\frac{t + \frac{2\pi}{3}}{2}\,sin\frac{t - \frac{2\pi}{3}}{2}}$$

$$= \frac{-1}{2sec\frac{2\pi}{3}\,sin\frac{2\pi}{3}}$$

$$= \frac{-1}{2(-2)\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2\sqrt{3}}$$

Verify : $\frac{d}{dx} sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}} = \frac{1}{|-2|\sqrt{4 - 1}} = \frac{1}{2\sqrt{3}}$

Exercise 5

Find points of discontinuity, if any, for following functions (1 to 4)

1. $f(x) = \begin{cases} \dfrac{x^3 - 27}{x - 3} & x \neq 3 \\ 5 & x = 3 \end{cases}$

2. $f(x) = \begin{cases} \dfrac{\sin(x-1)}{|x-1|} & x \neq 1 \\ 2 & x = 1 \end{cases}$

3. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x + 1} & x \neq -1 \\ -1 & x = -1 \end{cases}$

4. $f(x) = \begin{cases} \dfrac{e^{2x} - e^4}{e^x - e^2} & x \neq 2 \\ e^2 & x = 2 \end{cases}$

Find k, if following functions are continuous at given value of x : (5 to 8)

5. $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3} & x \neq 3 \\ k & x = 3 \end{cases}$ at $x = 3$

6. $f(x) = \begin{cases} kx^2 & x < 1 \\ x^2 + 1 & x \geq 1, \end{cases}$ at $x = 1$

7. $f(x) = \begin{cases} 2x + 3 & x < 2 \\ k & x = 2 \\ 3x + 1 & x > 2 \end{cases}$ at $x = 2$

8. $f(x) = \begin{cases} \cos x & 0 < x < \frac{\pi}{2} \\ k^2 - 4 & x = \frac{\pi}{2} \\ \sin x - 1 & x > \frac{\pi}{2} \end{cases}$ at $x = \frac{\pi}{2}$

Find a and b, if following functions are continuous (9 to 10) :

9. $f(x) = \begin{cases} a\sin x + b & 0 \leq x \leq \frac{\pi}{2} \\ \cos x & \frac{\pi}{2} < x \leq \pi \\ \tan x + b & \pi < x < \frac{3\pi}{2} \end{cases}$

10. $f(x) = \begin{cases} ax + b & 0 \leq x < 1 \\ 2x + 3 & 1 \leq x < 2 \\ x + a & x \geq 2 \end{cases}$

Find $\dfrac{dy}{dx}$ for following functions y where ever defined :

11. $y = \log_{10}(x^2 + 1)$

12. $y = \cot^{-1}\dfrac{2x}{1 - x^2}$ $\quad x \neq \pm 1$

13. $y = \sin(\log(\cos x))$

14. $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a, \; |x| < 1, \; |y| < 1$

15. $y = (\sin x)^{\sin x}$

16. $y = (\sin x - \cos x)^{\sin x - \cos x}$

17. $y = x^x + \left(x + \frac{1}{x}\right)^x$

18. $y = x^{\left(x + \frac{1}{x}\right)}$

19. $y = \cos(x^x) + (\tan x)^x$

20. $y = \sin^{-1}x + \sin^{-1}\sqrt{1 - x^2}, \; |x| < 1$

21. $y = \tan^{-1}x + \cot^{-1}x \;\; x \in R$

22. $x = (\cos t)^t \;\; y = (\sin t)^t \quad 0 < t < \frac{\pi}{2}$

23. Prove $\dfrac{d}{dx} e^{ax}\cos(bx + c) = re^{ax}\cos(bx + c + \alpha)$ where $r = \sqrt{a^2 + b^2}$, $\cos\alpha = \frac{a}{r}$, $\sin\alpha = \frac{b}{r}$

and $\dfrac{d^2}{dx^2} e^{ax}\cos(bx + c) = r^2 e^{ax}\cos(bx + c + 2\alpha)$

24. Find $\frac{d}{dx}\ tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $x \in R - \{0\}$

25. Find $\frac{d}{dx}\ tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$, $|x| < 1$

26. Find $\frac{d}{dx}\ tan^{-1}\sqrt{\frac{1+sinx}{1-sinx}}$ $\quad 0 < x < \frac{\pi}{2}$

27. If $y = (cos^{-1}x)^2$, prove $(1 - x^2)y_2 - xy_1 = 2$

28. If $y = sinpt$, $x = sint$ prove $(1 - x^2)y_2 - xy_1 + p^2y = 0$

29. If $y = e^{mtan^{-1}x}$, prove $(1 + x^2)y_2 + (2x - m)y_1 = 0$

30. If $2x = y^{\frac{1}{m}} + y^{-\frac{1}{m}}$ $(x \geq 1)$, prove $(x^2 - 1)y_2 + xy_1 = m^2y$

31. If $y = (x + \sqrt{x^2-1})^m$, prove $(x^2 - 1)y_2 + xy_1 = m^2y$

32. If $x^y = e^{x-y}$, prove $\frac{dy}{dx} = \frac{\log x}{(\log x+1)^2}$

33. If $y = e^{ax}\ sinbx$, prove $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

34. If $(a - bcosy)(a + bcosx) = a^2 - b^2$, prove $\frac{dy}{dx} = \frac{\sqrt{a^2-b^2}}{a+bcosx}$, $0 < x < \frac{\pi}{2}$

35. If $y = (tan^{-1}x)^2$, prove $(1 + x^2)^2y_2 + 2x(1 + x^2)y_1 = 2$

36. If $y = x\log\frac{x}{a+bx}$, prove $x^3y_2 = (xy_1 - y)^2$

37. If $x = asint - bcost$, $y = acost + bsint$, find y_2.

38. If $y = sin(sinx)$, prove $y_2 + tanx \cdot y_1 + ycos^2x = 0$

39. If $y = cos^{-1}\frac{3+5cosx}{5+3cosx}$, prove $\frac{dy}{dx} = \frac{4}{5+3cosx}$.

40. Find the derivative of $tan^{-1}\frac{x}{\sqrt{1-x^2}}$ w.r.t. $sin^{-1}(2x\sqrt{1-x^2})$. $\quad 0 < x < \frac{1}{\sqrt{2}}$

41. Find the derivative of $cos^{-1}\frac{1-x^2}{1+x^2}$ w.r.t. $sin^{-1}\frac{2x}{1+x^2}$.

42. Find $\left[\frac{d}{dx}(cosec^{-1}x)\right]_{x=-2}$ by defination.

43. Find $\frac{d}{dx}\left(sin^{-1}\frac{x}{\sqrt{1+x^2}}+cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$, $\quad x > 0$

44. Find $\frac{d}{dx}\ tan^{-1}\frac{4x}{1+21x^2}$, $\quad x > 0$

45. Find $\frac{d}{dx} tan^{-1} \frac{a+bx}{b-ax}$

46. Find $\frac{d}{dx} tan^{-1} \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ $|x| < 1$

47. Find $\frac{d}{dx} tan^{-1}(secx - tanx)$.

48. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A (1 mark)

(1) $\left[\frac{d}{dx} sec^{-1}x\right]_{x = -3}$ = ... ☐

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $-\frac{1}{\sqrt{x^2-1}}$ (c) $\frac{1}{6\sqrt{2}}$ (d) $\frac{-1}{6\sqrt{2}}$

(2) $\frac{d}{dx} x^x$ = ... $(x > 0)$ ☐

(a) x^{x-1} (b) x^x (c) 0 (d) $x^x(1 + \log x)$

(3) $\frac{d}{dx} (sin^{-1}x + cos^{-1}x)$ = ... $(|x| < 1)$ ☐

(a) 0 (b) $\frac{2}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1-x^2}}$ (d) does not exist

(4) $\frac{d}{dx} a^a$ = ... $(a > 0)$ ☐

(a) $a^a(1 + \log a)$ (b) 0 (c) a^a (d) does not exist

(5) $\frac{d}{dx} e^{5x}$ = ... ☐

(a) e^{5x} (b) $5e^{5x}$ (c) $5x\ e^{5x-1}$ (d) 0

(6) $\frac{d}{dx} \log |x|$ = ... $(x \neq 0)$ ☐

(a) $\frac{1}{|x|}$ (b) $\frac{1}{x}$ (c) does not exist (d) e^x

(7) $\frac{d}{dx} sin^3x$ = ... ☐

(a) $3sin^2x$ (b) $3cos^2x$ (c) $3sin^2x\ cosx$ (d) $-3cos^2x\ sinx$

(8) $\frac{d}{dx} tan^nx$ = ... ☐

(a) $ntan^{n-1}x$ (b) $ntan^{n-1}x\ sec^2x$ (c) $n\ sec^{2n}x$ (d) $ntan^{n-1}x\ sec^{n-1}x$

(9) If $f(x) = \begin{cases} ax + b & 1 \le x < 5 \\ 7x - 5 & 5 \le x < 10 \\ bx + 3a & x \ge 10 \end{cases}$

is continuous, $(a, b) =$ ......

(a) (5, 10) (b) (5, 5) (c) (10, 5) (d) (0, 0)

(10) If $f(x) = \begin{cases} \frac{x^2}{a} - a & x < a \\ 0 & x = a \\ a - \frac{x^2}{a} & x > a \end{cases}$ then...

(a) $\lim_{x \to a+} f(x) = a$ (b) $\lim_{x \to a} f(x) = -a$

(c) f is continuous at $x = a$ (d) f is differentiable at $x = a$

(11) If $f(x) = \begin{cases} x & x \in (0, 1) \\ 1 & x \ge 1 \end{cases}$

(a) f is continuous at $x = 1$ only (b) f is discontinuous at $x = 1$ only

(c) f is continuous on R^+ (d) f is not defined for $x = 1$

(12) $\frac{d}{dx} \frac{1}{\log |x|} =$ ....

(a) $\frac{1}{|x|}$ (b) $\frac{1}{(\log x)^2}$ (c) $-\frac{1}{x(\log |x|)^2}$ (d) e^x

(13) If $y = a sinx + b cosx$, $y^2 + (y_1)^2 =$ .... $(a^2 + b^2 \ne 0)$

(a) $a cosx - b sinx$ (b) $(a sinx - b cosx)^2$ (c) $a^2 + b^2$ (d) 0

(14) $\frac{d}{dx} (x^2 + sin^2 x)^3 =$ ....

(a) $3(x^2 + sin^2 x)$ (b) $3(x^2 + sin^2 x)^2 (2x + sin2x)$

(c) $2x + 2 sinx\ cosx$ (d) 0

(15) $\frac{d}{dx} \sqrt{x sinx} =$ .... $0 < x < \pi$

(a) $\frac{x sinx + cosx}{\sqrt{x sinx}}$ (b) $\frac{x cosx}{2\sqrt{x sinx}}$ (c) $\frac{x cosx + sinx}{2\sqrt{x sinx}}$ (d) $\frac{1}{2\sqrt{x sinx}}$

Section B (2 marks)

(16) $\frac{d}{dx} tan^{-1} \frac{1 - x}{1 + x} =$ ....

(a) $-\frac{1}{1 + x^2}$ (b) $\frac{1}{1 + x^2}$ (c) $\frac{1 + x}{1 - x}$ (d) $\frac{2}{1 + x^2}$

(17) $\frac{d}{dx}\ tan^{-1}\sqrt{\frac{1-cosx}{1+cosx}}$ = ... $\pi < x < 2\pi$

(a) $\frac{1}{1+cos^2x}$ (b) $-\frac{1}{1+cos^2x}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

(18) If $x = e^{tan^{-1}\frac{y-x^2}{x^2}}$, then $\frac{dy}{dx}$ = ...

(a) $2x\,(tan\,(\log x) + 1)$ (b) $2x\,(tan\,(\log x) + 1) + x^2\,sec\,(\log x)$

(c) $2x\,(tan\,(\log x) + 1) + x^2 sec\,(\log x)$ (d) 0

(19) $\frac{d}{dx}\ sin^{-1}\left(\frac{3x}{5}+\frac{4}{5}\sqrt{1-x^2}\right)$ = ... $(0 < x < \frac{3}{5})$

(a) $\frac{3}{5} + \frac{1}{\sqrt{1-x^2}}$ (b) $\frac{4}{5}\ \frac{1}{\sqrt{1-x^2}}$ (c) $-\frac{1}{\sqrt{1-x^2}}$ (d) $\frac{1}{\sqrt{1-x^2}}$

(20) $\frac{d}{dx}\ tan^{-1}\left(\frac{x+a}{1-xa}\right)$ = ... $(x,\ a \in R^+,\ xa > 1)$

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1+a^2}$ (c) $\frac{1}{1+x^2} + \frac{1}{1+a^2}$ (d) $\frac{1}{1+x^2a^2}$

(21) If $f(x) = \log_7(\log_3 x)$, then $f'(x)$ = ...

(a) $\frac{1}{x\log 7\ \log 3}$ (b) $\frac{1}{\log 3\ \log x}$ (c) $\frac{1}{x\ \log x\ \log 7}$ (d) $\frac{1}{x\log x}$

(22) $\frac{d}{dx}\ x\,|\,x\,|$ = ... $(x < 0)$

(a) $2x$ (b) $-2x$ (c) $|\,x\,|$ (d) 0

(23) If $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$, then $\frac{dy}{dx}$ = ...

(a) $\frac{2t^2}{1-t^2}$ (b) $\frac{2t}{1+t^2}$ (c) $2t$ (d) $\frac{-2t}{1-t^2}$

(24) $\frac{d}{dx}\ e^{x\log x}$ = ...

(a) $x^x(1 + \log x)$ (b) x^x (c) $1 + \log x$ (d) x^{x-1}

(25) $\frac{d}{dx}\ \frac{tan^{-1}x}{1+tan^{-1}x}$ w.r.t. $tan^{-1}x$ = ...

(a) $\frac{1}{1+tan^{-1}x}$ (b) $\frac{1}{(1+tan^{-1}x)^2}$ (c) $\frac{1}{1+x^2}$ (d) $\frac{-1}{1+x^2}$

Section C (3 marks)

(26) If $x = at^2$, $y = 2at$, then $\frac{d^2y}{dx^2}$ = ...

(a) $\frac{-1}{t^2}$ (b) $\frac{1}{t^2}$ (c) $\frac{-1}{2at^3}$ (d) $\frac{1}{2at^3}$

(27) $\frac{d}{dx} \cot^{-1} \frac{\sqrt{1+x^2}-1}{x} = \dots\dots \quad (x \in R - \{0\})$

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{2(1+x^2)}$ (c) $\frac{2}{1+x^2}$ (d) $-\frac{1}{1+x^2}$

(28) $\frac{d^2x}{dy^2} = \dots\dots$

(a) $\frac{1}{\frac{d^2y}{dx^2}}$ (b) $\frac{1}{\left(\frac{dy}{dx}\right)^2}$ (c) $-\frac{1}{\left(\frac{dy}{dx}\right)^2}$ (d) $-\frac{1}{\left(\frac{dy}{dx}\right)^3} \frac{d^2y}{dx^2}$

(29) For the curve $f(x) = (x-3)^2$, applying mean value theorem on [2, 4] the tangent at ...... is parallel to the chord joining A(2, 1) and B(4, 1).

(a) (1, 0) (b) (4, 3) (c) (2, 3) (d) (3, 0)

(30) The value of c for the mean-value theorem for $f(x) = x^3$ in $[-1, 1]$ is ....

(a) $\pm\frac{1}{\sqrt{3}}$ (b) $\pm\sqrt{3}$ (c) ± 1 (d) 0

(31) If we apply the Rolle's theorem to $f(x) = e^x \sin x \quad x \in [0, \pi]$, then $c = \dots\dots$.

(a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{4}$ (c) $\frac{\pi}{4}$ (d) $\frac{7\pi}{4}$

(32) If we apply the Rolle's theorem to $f(x) = x^3 - 4x$, $x \in [0, 2]$, then $c = \dots\dots$.

(a) $\sqrt{3}$ (b) 2 (c) $\frac{2}{\sqrt{3}}$ (d) -2

Section D (4 marks)

(33) If $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$, then...

(a) $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$ (b) $(x^2-4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2-4)$

(c) $(x^2+4)\left(\frac{dy}{dx}\right)^2 = 1$ (d) $(x^2+4)\left(\frac{dy}{dx}\right)^2 = y^2+4$

(34) $\frac{d}{dx} \tan^{-1} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^2}+\sqrt{1+x^2}} = \dots\dots \quad |x| < 1$

(a) $\frac{1}{\sqrt{1-x^4}}$ (b) $\frac{-x}{\sqrt{1-x^4}}$ (c) $\frac{1}{2\sqrt{1-x^4}}$ (d) $\frac{x^2}{1-x^4}$

(35) $\frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right) = \dots\dots \quad (a > 0)$

(a) $\frac{1}{\sqrt{a^2-x^2}}$ (b) $\sqrt{a^2-x^2}$ (c) $\sqrt{x^2-a^2}$ (d) $\sqrt{x^2+a^2}$

(36) Conditions of Mean Value Theorem are not applicable to ...... in $[-1, 1]$.

(a) $f(x) = |x|$ (b) $f(x) = x^3$ (c) $f(x) = \sin x$ (d) $f(x) = x^2$

(37) For $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$ the value of c for mean-value theorem and for $f(x) = x^2 - 4x + 3$ for Roll's theorem are .... ☐

(a) $\sqrt{3}$, 1 (b) 2, 1 (c) $\sqrt{3}$, 2 (d) 2, $\sqrt{3}$

(38) If the tangent to the curve $y = x \log x$ at $(c, f(x))$ is parallel to the line-segment joining A(1, 0) and B(e, e), then c = ... ☐

(a) $\frac{e-1}{e}$ (b) $\log \frac{e-1}{e}$ (c) $e^{\frac{1}{1-e}}$ (d) $e^{\frac{1}{e-1}}$

(39) If we apply the mean value theorem to $f(x) = 2\sin x + \sin 2x$, then c = ... ☐

(a) π (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

(40) If we apply the mean value theorem to $f(x) = \begin{cases} 2 + x^3 & x \le 1 \\ 3x & x > 1 \end{cases}$ $x \in [-1, 2]$ ☐

then c = ...

(a) 2 (b) 0 (c) 1 (d) $\frac{\sqrt{5}}{3}$

*

Summary

We have studied the following points in this chapter :

1. Continuous functions
2. Algebra of continuous functions
3. Differentiation and continuity
4. Chain rule
5. Rules for derivative of inverse function
6. Derivative of Implicit function
7. Derivative of parametric function
8. Logarithmic differentiation
9. Second order Derivative
10. Mean value theorems

Prehistory

Excavations at Harappa, Mohenjo-daro and other sites of the Indus Valley Civilization have uncovered evidence of the use of "practical mathematics". The people of the IVC manufactured bricks whose dimensions were in the proportion 4:2:1, considered favourable for the stability of a brick structure. They used a standardized system of weights based on the ratios: 1/20, 1/10, 1/5, 1/2, 1, 2, 5, 10, 20, 50, 100, 200, and 500, with the unit weight equal to approximately 28 grams (and approximately equal to the English ounce or Greek uncia). They mass produced weights in regular geometrical shapes, which included hexahedra, barrels, cones, and cylinders, thereby demonstrating knowledge of basic geometry.

The inhabitants of Indus civilization also tried to standardize measurement of length to a high degree of accuracy. They designed a ruler—the Mohenjo-daro ruler—whose unit of length (approximately 1.32 inches or 3.4 centimetres) was divided into ten equal parts. Bricks manufactured in ancient Mohenjo-daro often had dimensions that were integral multiples of this unit of length.