INDEFINITE INTEGRATION 6

What we know is not much, what we do not know is immense.
(Allegedly his last words)

– *Laplace*

A mathematics teacher is midwife to ideas.

– *George Polya*

6.1 Introduction

In the chapter on derivatives, we have already learnt about the differentiability of a function on some interval I. If a function is differentiable in an interval I, we know how to find its unique derivative f' at each point on I. Now, we shall study an operation which is 'inverse' to differentiation. For example we know that the derivative of x^3 with respect to x is $3x^2$. Now if we raise the question, derivative of which function or functions is $3x^2$? Then, it is difficult to find the answer. It is a question of an operation inverse to the operation of differentiation.

Let us frame a general question, "Is there a function whose derivative a given function can be and if there is such a function, how to find it ?" The process of finding answer to this question is called 'antiderivation'. It is possible that this question has no answer or it may have more than one answer. For example, (i) $\frac{d}{dx}(x^3) = 3x^2$, $\frac{d}{dx}(x^3 - 15) = 3x^2$ and in general $\frac{d}{dx}(x^3 + c) = 3x^2$, where c is any constant. (ii) $\frac{d}{dx}(sinx) = cosx$, $\frac{d}{dx}(sinx - 3) = cosx$. In general $\frac{d}{dx}(sinx + c) = cosx$.

Thus, antiderivatives of the above functions are not unique. Actually, there exist infinitely many antiderivatives of these functions which can be obtained by choosing c, from the set of real numbers. For this reason, such a constant is called an arbitrary constant.

6.2 Definition

If we can find a function g defined on an interval I such that $\frac{d}{dx}(g(x)) = f(x)$, $\forall x \in$ I, then $g(x)$ is called a **primitive** or **antiderivative** or **indefinite integral** of $f(x)$. It is denoted by $\int f(x)dx$. $\int f(x)dx$ is called an indefinite integral of $f(x)$ with respect to x. The process (operation) of finding $g(x)$, given $f(x)$ is called indefinite integration. This 'indefiniteness' is upto arbitrary constant.

Thus, the question whether we can find primitive of f is not easy to answer. There are some sufficient conditions such as continuous functions and monotonic functions have primitives. $\frac{sinx}{x}$ is continuous, $\int \frac{sinx}{x} dx$ is defined, but cannot be expressed in terms of known elementary functions. Similarly, $\int \sqrt{secx}\, dx$ and $\int \sqrt{x^3+1}\, dx$ cannot be expressed as a known function.

In $\int f(x)dx$, $\int ....dx$ indicates the process of integration with respect to x. $\int f(x)dx$ denotes, integral of $f(x)$ with respect to x and in $\int f(x)dx$, $f(x)$ is called integrand.

6.3 Some Theorems on Antiderivative :

Theorem 6.1 : If f and g are differentiable on (a, b) and if $f'(x) = g'(x)$, $\forall x \in (a, b)$, then $f(x) = g(x) + c$, where c is a constant.

Proof : Let $h(x) = f(x) - g(x)$, $x \in (a, b)$.

f and g are differentiable on (a, b) and hence f and g are continuous on (a, b).

$\therefore$ If $x_1, x_2 \in (a, b)$, $x_1 < x_2$, then h is continuous on $[x_1, x_2]$.

Now, h is differentiable on (x_1, x_2) as $[x_1, x_2] \subset (a, b)$.

$\therefore$ By mean value theorem,

$$\frac{h(x_2) - h(x_1)}{x_2 - x_1} = h'(c) \text{ for some } c \in (x_1, x_2).$$

$\therefore$ $h(x_2) - h(x_1) = h'(c)(x_2 - x_1)$. **(i)**

Now $c \in (x_1, x_2) \Rightarrow c \in (a, b)$

But it is given that $\forall x \in (a, b)$, $f'(x) = g'(x)$.

$\therefore$ $f'(c) = g'(c)$

$\therefore$ $f'(c) - g'(c) = 0$

$\therefore$ $h'(c) = 0$ **$(h(x) = f(x) - g(x) \Rightarrow h'(x) = f'(x) - g'(x))$**

$\therefore$ $h(x_2) - h(x_1) = 0 \quad \forall x_1, x_2 \in (a, b)$ **(by (i))**

$\therefore$ $h(x_1) = h(x_2)$

$\therefore$ $f(x_1) - g(x_1) = f(x_2) - g(x_2), \quad \forall x_1, x_2 \in (a, b)$

$\therefore$ $f - g$ is a constant function on (a, b).

$\therefore$ $f(x) - g(x) = c$, where $c \in R$ is a constant.

$\therefore$ $f(x) = g(x) + c, \quad \forall x \in (a, b)$

General Antiderivative : If $\frac{d}{dx}(f(x)) = \frac{d}{dx}(g(x)) = h(x)$, then $\int h(x)dx = f(x)$ and $\int h(x)dx = g(x)$.

But $f(x) = g(x) + c$. So $\int h(x)dx = f(x) = g(x) + c$. Here $g(x)$ is a differentiable function on (a, b) with $\frac{d}{dx}(g(x)) = \frac{d}{dx}f(x) = h(x)$. Hence if one integral of $h(x)$ is $g(x)$, any other integral of $h(x)$ is $g(x) + c$. Also if $\frac{d}{dx}(g(x)) = h(x)$, then $\frac{d}{dx}[g(x) + c] = \frac{d}{dx}g(x) = h(x)$.

Thus $g(x) + c$ is also an integral of $f(x)$.

Thus, if one primitive of $h(x)$ is $g(x)$, then all its primitives are given by $g(x) + c$, where c is a constant. As c is any constant, it is called an arbitrary constant.

Let us perform the operation of differentiation and integration successively in any order.

By definition of antiderivative, we know that,

$$\frac{d}{dx}g(x) = f(x), \forall x \in I \Leftrightarrow \int f(x)dx = g(x) + c.$$

Now, $\frac{d}{dx}[\int f(x)dx] = \frac{d}{dx}[g(x) + c] = f(x)$.

$\therefore$ If we first integrate $f(x)$ and then differentiate the integral, we get the same function $f(x)$ as a result.

But, $\int \left[\frac{d}{dx} g(x)\right] dx = \int f(x)dx = g(x) + c.$

If we first differentiate the function $g(x)$ and then integrate its derivative, we get $g(x) + c$.

Theorem 6.2 : If f and g are integrable on (a, b), then $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$.

Proof : $\frac{d}{dx}\left[\int f(x)dx + \int g(x)dx\right] = \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx$

$= f(x) + g(x)$

$\therefore$ Using the definition of antiderivative,

$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

In general if $f_1, f_2, f_3, ..., f_n$ are integrable over an interval, then

$\int [f_1(x) + f_2(x) + ... + f_n(x)] = \int f_1(x)dx + \int f_2(x)dx + ... + \int f_n(x)dx.$

Theorem 6.3 : If f is an integrable function on (a, b) and $k \in R$, then $\int kf(x)dx = k\int f(x)dx$.

Proof : $\frac{d}{dx}[k\int f(x)dx] = k\frac{d}{dx}\int f(x)dx$

$= kf(x)$

$\therefore$ Using the definition of antiderivative,

$\int kf(x)dx = k\int f(x)dx.$

Corollary 1 : If f and g are integrable functions in (a, b), then

$$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$$

Proof : $\int (f(x) - g(x))dx = \int (f(x)dx + (-1)g(x))dx$

$= \int f(x)dx + \int (-1)g(x)dx$

$= \int f(x)dx + (-1)\int g(x)dx$

$= \int f(x)dx - \int g(x)dx$

Thus, $\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx$

In general, $\int [k_1f_1(x) + k_2f_2(x) + ... + k_nf_n(x)]dx$

$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + ... + \int k_nf_n(x)dx$

Theorem 6.2, 6.3 and corollary 1 are known as working rules for integration.

6.4 Standard Integrals

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \in R - \{-1\}, x \in R^+$.

$\frac{x^{n+1}}{n+1}$ is differentiable for all $x \in R^+$ and $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}[(n + 1)x^n] = x^n$

$\therefore$ By the definition of antiderivative, $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad \forall x \in R^+$.

(Also let us remember that if $g(x)$ is one primitive then $g(x) + c$ is the general primitive.)

Thus, for $n = 0$, $\int x^0\,dx = \frac{x^{0+1}}{0+1} + c = x + c$

$\therefore$ $\int dx = x + c$

(2) $\int \frac{1}{x} dx = \log|x| + c, x \in R - \{0\}$.

$\log|x|$ is a differentiable function, $\forall x \in R - \{0\}$ and if $x > 0$, $\frac{d}{dx}(\log|x|) = \frac{d}{dx}\log x = \frac{1}{x}$.

If $x < 0$, $\frac{d}{dx}\log|x| = \frac{d}{dx}\log(-x) = \frac{-1}{-x} = \frac{1}{x}$

$\therefore$ $\frac{d}{dx}\log|x| = \frac{1}{x} \quad \forall x \in R - \{0\}$

$\therefore$ By the definition of antiderivative,

$\int \frac{1}{x} dx = \log|x| + c, \quad \forall x \in R - \{0\}$.

We write $\int \frac{dx}{x} = \log|x| + c, x \neq 0$.

(3) $\int cosx\, dx = sinx + c, \forall x \in R$

sin is a differentiable function $\forall x \in R$ and $\frac{d}{dx}(sinx) = cosx, \quad \forall x \in R$

$\therefore$ By the definition of antiderivative,

$\int cosx\, dx = sinx + c, \quad \forall x \in R$

In the same way, we can prove that

(4) $\int sinx\, dx = -cosx + c, \forall x \in R$

(5) $\int sec^2x\, dx = tanx + c, x \neq (2k - 1)\frac{\pi}{2}, k \in Z$

tan is differentiable on any interval not containing $(2k - 1)\frac{\pi}{2}$, $k \in Z$ and $\frac{d}{dx}(tanx) = sec^2x$.

$\therefore$ By the definition of antiderivative, $\int sec^2x\, dx = tanx + c, x \neq (2k - 1)\frac{\pi}{2}, k \in Z$

In the same way, we can prove that

(6) $\int cosec^2x\, dx = -cotx + c, x \neq k\pi, k \in Z$

(7) $\int secx\ tanx\, dx = secx + c, x \neq (2k - 1)\frac{\pi}{2}, k \in Z$

(8) $\int cosecx\ cotx\, dx = -cosecx + c, x \neq k\pi, k \in Z$

(9) $\int a^x\, dx = \frac{a^x}{\log_e a} + c, a \in R^+ - \{1\}, x \in R$

$\frac{a^x}{\log_e a}$ is differentiable $\forall x \in R$ and $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = \frac{1}{\log_e a}(a^x \log_e a) = a^x, \quad \forall x \in R$

$\therefore$ By the definition of antiderivative, $\int a^x\, dx = \frac{a^x}{\log_e a} + c, a \in R^+ - \{1\}$.

Now, for $a = e$

$\int e^x\, dx = \frac{e^x}{\log_e e} + c$

$\therefore$ $\int e^x\, dx = e^x + c, \quad \forall x \in R$.

(10) $\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + c, \quad a \in R - \{0\}, x \in R$

$= -\frac{1}{a} cot^{-1}\left(\frac{x}{a}\right) + c', \quad a \in R - \{0\}, x \in R$

$tan^{-1}\left(\frac{x}{a}\right)$ is differentiable for $\forall x \in R$ and for any non-zero constant a.

$\therefore$ $\frac{1}{a}\, tan^{-1}\left(\frac{x}{a}\right)$ is differentiable for $\forall a \in R - \{0\}$ and

$$\frac{d}{dx}\left[\frac{1}{a}\, tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{a} \cdot \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a} = \frac{1}{x^2+a^2}$$

$\therefore$ By the definition of antiderivative, $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\, tan^{-1}\left(\frac{x}{a}\right) + c, \forall x \in R$

Thus, $\frac{1}{a} tan^{-1}\frac{x}{a}$ and $-\frac{1}{a} cot^{-1}\frac{x}{a}$ both can be taken as integrals of $\frac{1}{x^2+a^2}$.

Let us try to understand the reason behind this.

Let $f(x) = \frac{1}{a} tan^{-1}\frac{x}{a}$ and $g(x) = -\frac{1}{a} cot^{-1}\frac{x}{a}$.

Now, we know that $tan^{-1}\frac{x}{a} + cot^{-1}\frac{x}{a} = \frac{\pi}{2}$

$\therefore$ $\frac{1}{a} tan^{-1}\frac{x}{a} + \frac{1}{a} cot^{-1}\frac{x}{a} = \frac{\pi}{2a}$

$\therefore$ $f(x) - g(x) = \frac{\pi}{2a}$

$\therefore$ $f(x) = g(x) + \frac{\pi}{2a}$.

$\therefore$ $\frac{d}{dx} f(x) = \frac{d}{dx} g(x)$.

As antiderivative is not unique, $\int h(x)dx = g(x)$ and $\int h(x)dx = f(x)$ does not give $f(x) = g(x)$. We can say that there is a constant c such that $f(x) = g(x) + c$.

(11) $\int \frac{dx}{x^2-a^2}\, dx = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c, a \in R - \{0\}$ **(on any interval not containing $-a$ and a)**

On any interval not containing $-a$ and a, $\frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|$ is differentiable and

$$\frac{d}{dx}\left(\frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|\right) = \frac{1}{2a}\frac{d}{dx} [\log|x-a| - \log|x+a|]$$

$$= \frac{1}{2a} \cdot \left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$= \frac{1}{2a} \cdot \left[\frac{x+a-x+a}{(x-a)(x+a)}\right]$$

$$= \frac{1}{2a} \cdot \left(\frac{2a}{x^2-a^2}\right)$$

$$= \frac{1}{x^2-a^2}$$

$\therefore$ Using the definition of antiderivative, $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c, a \in R - \{0\}$

(12) $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a} \log \left|\frac{x+a}{x-a}\right| + c, a \in R - \{0\}$ **(on any interval not containing $-a$ and a)**

We have, $\int \frac{1}{a^2-x^2}\, dx = -1 \int \frac{1}{x^2-a^2}\, dx$

$$= -\frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c$$

$$= \frac{1}{2a} \log \left|\frac{x+a}{x-a}\right| + c$$

(13) $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} + c, \quad x \in (-a, a), a > 0.$

$$= -\cos^{-1} \frac{x}{a} + c', \quad x \in (-a, a), a > 0.$$

$\sin^{-1}\left(\frac{x}{a}\right)$ is a differentiable function for $x \in (-a, a), a > 0$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{|a|}{\sqrt{a^2 - x^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2 - x^2}} \qquad (a > 0, |a| = a)$$

$\therefore$ Using the definition of antiderivative. $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} + c, \ x \in (-a, a), a > 0.$

As shown in (10) we have $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = -\cos^{-1} \frac{x}{a} + c', \ x \in (-a, a)$

Also if $a < 0$, then $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = -\sin^{-1} \frac{x}{a} + c = \cos^{-1} \frac{x}{a} + c'.$ (as $|a| = -a$)

We shall usually use the formula for $a > 0$.

(14) $\int \frac{1}{|x|\sqrt{x^2 - a^2}}\, dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c, \quad |x| > |a| > 0.$

$$= -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + c', \quad |x| > |a| > 0.$$

If $a \in R - \{0\}$ and $|x| > |a|$, $\frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right)$ is differentiable and

$$\frac{d}{dx}\left(\frac{1}{a} \sec^{-1} \frac{x}{a}\right) = \frac{1}{a} \cdot \frac{1}{\left|\frac{x}{a}\right| \sqrt{\frac{x^2}{a^2} - 1}} \cdot \frac{1}{a}$$

$$= \frac{1}{a^2} \cdot \frac{|a|^2}{|x|\sqrt{x^2 - a^2}}$$

$$= \frac{1}{a^2} \cdot \frac{a^2}{|x|\sqrt{x^2 - a^2}}$$

$$= \frac{1}{|x|\sqrt{x^2 - a^2}}$$

$\therefore$ Using the definition of antiderivative, $\int \frac{1}{|x|\sqrt{x^2 - a^2}}\, dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c, \ (|x| > |a| > 0)$

As shown in (10) we can write $\int \frac{1}{|x|\sqrt{x^2 - a^2}}\, dx = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + c'$

(15) $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \log \left| x + \sqrt{x^2 \pm a^2} \right| + c, \quad \forall x \in R.$

$$\frac{d}{dx}\left(\log \left| x + \sqrt{x^2 \pm a^2} \right|\right) = \frac{1}{x + \sqrt{x^2 \pm a^2}} \frac{d}{dx}\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$= \frac{1}{x + \sqrt{x^2 \pm a^2}} \left(1 + \frac{2x}{2\sqrt{x^2 \pm a^2}}\right)$$

$$= \frac{1}{x + \sqrt{x^2 \pm a^2}} \left(\frac{\sqrt{x^2 \pm a^2} + x}{\sqrt{x^2 \pm a^2}}\right)$$

$$= \frac{1}{\sqrt{x^2 \pm a^2}}$$

$\therefore$ Using the definition of antiderivative, $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log \left| x + \sqrt{x^2 \pm a^2} \right| + c, \forall x \in R$

(**Note :** For existence of $\frac{dx}{\sqrt{x^2 - a^2}}$, it is necessary that $|x| > |a|$.)

Generally, if $g(x)$ is any primitive of $f(x)$, we will not write $\int f(x)dx = g(x) + c$. But instead, we will write $\int f(x)dx = g(x)$ assuming that c is included in $g(x)$. According to this, in an equation like $\int f(x)dx = \int g(x)dx + \int h(x)dx$, there is no need to write c. It is included in the symbol $\int .....dx$. But it is necessary to write $\int x^2 dx = \frac{x^3}{3} + c$. Here $\frac{x^3}{3}$ is not the general integral. It is one integral.

Thus, we may introduce c when all symbols $\int .....dx$ are removed after carrying out integration.

Again, it is not necessary to write $\int x^2 dx + \int \frac{x^3}{3} dx = \frac{x^3}{3} + c_1 + \frac{x^4}{4} + c_2$ as $c_1 + c_2$ is also an arbitrary constant. Thus, we can write $\int x^2 dx + \int x^3 dx = \frac{x^3}{3} + \frac{x^4}{4} + c$.

For the following examples, we will assume that integral is defined on some appropriate domain of R. We use symbol I for an integral.

Example 1 : Obtain the integral of the following functions *w.r.t. x*.

(1) $x^{\frac{5}{2}} + 4 \cdot 3^x - \frac{1}{x}$ **(2)** $\frac{(2x+1)^3}{\sqrt{x}}, (x > 0)$ **(3)** $\frac{x}{a} + \frac{a}{x} + x^a + a^x$ **(4)** $\frac{1}{1 + \cos 2x}$

(5) $\frac{1}{9 - x^2}, x^2 \neq 9$ **(6)** $\frac{1}{\sqrt{x^2 - 4}}, |x| > 2$

Solution : (1) $\text{I} = \int \left(x^{\frac{5}{2}} + 4 \cdot 3^x - \frac{1}{x}\right) dx$

$$= \int x^{\frac{5}{2}} dx + 4\int 3^x dx - \int \frac{1}{x} dx$$

$$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 4 \cdot \frac{3^x}{\log_e 3} - \log|x| + c$$

$$= \frac{2}{7} x^{\frac{7}{2}} + \frac{4 \cdot 3^x}{\log_e 3} - \log|x| + c$$

(2) I $= \int \frac{(2x+1)^3}{\sqrt{x}}\, dx = \int \frac{8x^3+1+12x^2+6x}{\sqrt{x}}\, dx$

$$= \int \left(\frac{8x^3}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} + \frac{12x^2}{x^{\frac{1}{2}}} + \frac{6x}{x^{\frac{1}{2}}} \right) dx$$

$$= 8 \int x^{\frac{5}{2}} dx + \int x^{-\frac{1}{2}} dx + 12 \int x^{\frac{3}{2}} dx + 6 \int x^{\frac{1}{2}} dx$$

$$= 8 \cdot \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 12 \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 6 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{16}{7} x^{\frac{7}{2}} + 2x^{\frac{1}{2}} + \frac{24}{5} x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + c$$

(3) I $= \int \left(\frac{x}{a} + \frac{a}{x} + x^a + a^x \right) dx = \frac{1}{a} \int x\, dx + a \int \frac{1}{x}\, dx + \int x^a dx + \int a^x dx$

$$= \frac{1}{a} \frac{x^2}{2} + a \log|x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log_e a} + c$$

$$= \frac{x^2}{2a} + a \log|x| + \frac{x^{a+1}}{a+1} + \frac{a^x}{\log_e a} + c$$

(4) I $= \int \frac{1}{1+\cos 2x}\, dx = \int \frac{1}{2\cos^2 x}\, dx$

$$= \frac{1}{2} \int \sec^2 x\, dx$$

$$= \frac{1}{2} \tan x + c$$

(5) I $= \int \frac{1}{9-x^2}\, dx = \int \frac{1}{(3)^2-(x)^2}\, dx$

$$= \frac{1}{2(3)} \log \left| \frac{x+3}{x-3} \right| + c$$

$$= \frac{1}{6} \log \left| \frac{x+3}{x-3} \right| + c$$

(6) I $= \int \frac{1}{\sqrt{x^2-4}}\, dx = \int \frac{1}{\sqrt{x^2-2^2}}\, dx$

$$= \log \left| x + \sqrt{(x)^2-(2)^2} \right| + c$$

$$= \log \left| x + \sqrt{x^2-4} \right| + c$$

Example 2 : Evaluate the following :

(1) $\int \frac{dx}{4x^2+9}$ (2) $\int \frac{dx}{9x^2-25}$, $x^2 \neq \frac{25}{9}$ (3) $\int \frac{(x^4+x^2+3)dx}{2(x^2+1)}$ (4) $\int \frac{(x^2+5)dx}{x^2-5}$, $x^2 \neq 5$

(5) $\int \frac{\sin x\, dx}{1+\sin x}$ (6) $\int \sec^2 x \cdot \operatorname{cosec}^2 x\, dx$

Solution : (1) I $= \int \frac{1}{4x^2+9}\, dx$

$$= \frac{1}{4} \int \frac{1}{x^2+\frac{9}{4}}\, dx$$

$$= \frac{1}{4}\int \frac{1}{(x^2)+\left(\frac{3}{2}\right)^2}\,dx$$

$$= \frac{1}{4}\,\frac{1}{\left(\frac{3}{2}\right)}\;tan^{-1}\left(\frac{x}{\frac{3}{2}}\right) + c$$

$$= \frac{1}{6}\,tan^{-1}\left(\frac{2x}{3}\right) + c$$

(2) $I = \int \frac{1}{9x^2-25}\,dx$

$$= \frac{1}{9}\int \frac{1}{x^2-\frac{25}{9}}\,dx$$

$$= \frac{1}{9}\int \frac{1}{(x)^2-\left(\frac{5}{3}\right)^2}\,dx$$

$$= \frac{1}{9}\,\frac{1}{2\left(\frac{5}{3}\right)}\,\log\left|\frac{x-\frac{5}{3}}{x+\frac{5}{3}}\right| + c$$

$$= \frac{1}{30}\log\left|\frac{3x-5}{3x+5}\right| + c$$

(3) $I = \int \frac{x^4+x^2+3}{2(x^2+1)}\,dx$

$$= \int \frac{x^2(x^2+1)+3}{2(x^2+1)}\,dx$$

$$= \frac{1}{2}\int\left(x^2+\frac{3}{x^2+1}\right)dx$$

$$= \frac{1}{2}\int x^2dx + \frac{3}{2}\int\frac{1}{x^2+1^2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right] + \frac{3}{2}\,tan^{-1}x + c$$

$$= \frac{x^3}{6} + \frac{3}{2}\,tan^{-1}x + c$$

(4) $I = \int \frac{x^2+5}{x^2-5}\,dx,\ x^2 \neq 5$

$$= \int \frac{(x^2-5)+10}{x^2-5}\,dx$$

$$= \int\left(1+\frac{10}{x^2-5}\right)dx$$

$$= \int dx + 10\int\frac{1}{(x)^2-\left(\sqrt{5}\right)^2}\,dx$$

$$= x + \frac{10}{2\sqrt{5}}\,\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$$

$$= x + \sqrt{5}\,\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + c$$

(5) $I = \int \frac{sinx}{1+sinx}\,dx$

$$= \int \frac{sinx}{1+sinx}\times\frac{1-sinx}{1-sinx}\,dx$$

$$= \int \frac{sinx - sin^2x}{1-sin^2x}\,dx$$

$$= \int \frac{sinx - sin^2x}{cos^2x}\,dx$$

$$= \int\left(\frac{sinx}{cos^2x} - \frac{sin^2x}{cos^2x}\right)dx$$

$$= \int (secx\ tanx - tan^2x)\,dx$$

$$= \int secx\ tanx\ dx - \int (sec^2x - 1)\,dx$$

$$= \int secx\ tanx\ dx - \int sec^2x\ dx + \int 1\,dx$$

$$= secx - tanx + x + c$$

(6) $I = \int sec^2x \cdot cosec^2x$

$$= \int \frac{1}{cos^2x\,sin^2x}\,dx$$

$$= \int \frac{sin^2x + cos^2x}{sin^2x \cdot cos^2x}\,dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x + c$$

Example 3 : Evaluate : $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\, dx$

Solution : $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\, dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{(\cos x - \cos \alpha)}\, dx$$

$$= 2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha}\, dx$$

$$= 2\int (\cos x + \cos \alpha)\, dx$$

$$= 2\int \cos x\, dx + 2\cos \alpha \int 1\, dx$$

$$= 2\sin x + 2\cos \alpha \cdot x + c$$

$$= 2(\sin x + x\cos \alpha) + c$$

Example 4 : Evaluate : $\int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}}\, dx, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution : $I = \int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}}\, dx$

$$= \int \tan^{-1} \sqrt{\frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}}\, dx$$

$-\frac{\pi}{2} < x < \frac{\pi}{2}$. So, $-\frac{\pi}{4} < -\frac{x}{2} < \frac{\pi}{4}$.

$\therefore \quad 0 < \frac{\pi}{4} - \frac{x}{2} < \frac{\pi}{2}$.

$\therefore \quad I = \int \tan^{-1} \left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) dx \qquad \left(0 < \left(\frac{\pi}{4} - \frac{x}{2}\right) < \frac{\pi}{2}\right)$

$$= \int \left(\frac{\pi}{4} - \frac{x}{2}\right) dx$$

$$= \frac{\pi}{4}x - \frac{x^2}{4} + c$$

Example 5 : If $f'(x) = 3x^2 - \frac{2}{x^3}$ and $f(1) = 4$, find $f(x)$.

Solution : We have, $f'(x) = 3x^2 - \frac{2}{x^3}$

$\therefore \quad f(x) = \int (3x^2 - 2x^{-3})\, dx$

$\therefore \quad f(x) = 3\frac{x^3}{3} - 2\frac{x^{-2}}{-2} + c$

$\therefore \quad f(x) = x^3 + \frac{1}{x^2} + c$ **(i)**

Now, $f(1) = 1^3 + \frac{1}{1^2} + c$

$\therefore \quad 4 = 1 + 1 + c$

$\therefore \quad c = 2$ **($f(1) = 4$)**

$\therefore \quad f(x) = x^3 + \frac{1}{x^2} + 2$ **(Substituting $c = 2$ in (i))**

Exercise 6.1

Integrate the following functions *w.r.t.* x considering them well defined and integrable over proper domain :

1. $3x^2 + 5x - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}$
2. $\frac{5x^3 + x^2 + 2}{\sqrt{x}}$
3. $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3$
4. $(ax^2 + bx + c)\sqrt{x}$
5. $x^e + e^x + e^e$
6. $e^{a \log x} + e^{x \log a}$
7. $\frac{x^3 - 8}{x^2 - 2x}$
8. $2^x + \frac{1}{\sqrt{x^2 - 9}}$
9. $\frac{2x^3 + 18x - 1}{x^2 + 9}$
10. $\frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x}$
11. $\frac{x^4 + x^2 + 1}{x^2 + x + 1}$
12. $\frac{x^6 + 2}{x^2 + 1}$
13. $\frac{x^4 + 1}{x^2 + 1}$
14. $3\sin x + 5\cos x + \frac{7}{\cos^2 x} - \frac{4}{\sin^2 x} + \tan^2 x$
15. $\frac{2 + 3\cos x}{\sin^2 x}$
16. $(2\tan x - 3\cot x)^2$
17. $\frac{\cos 2x}{\sin^2 2x}$
18. $\frac{\cos x}{\cos x - 1}$
19. $\frac{1}{1 + \cos x}$
20. $\frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}$
21. $\frac{\cot x}{\operatorname{cosec} x - \cot x}$
22. $\frac{\tan x}{\sec x + \tan x}$
23. $(a\tan x + b\cot x)^2$
24. $\frac{x^2}{x^2 - 3}$
25. If $f'(x) = 8x^3 - 2x$, $f(2) = 8$, then find $f(x)$.

*

6.5 Method of Substitution for Integration

If the integrand $f(x)$ is in one of the standard forms or it can be put in one such form, it can be easily integrated. But if the integrand $f(x)$ is not in one of the standard forms or cannot be easily converted in one such form, then we may use a very useful method of substition.

In this method $\int f(x)dx$ is converted into $\int g(t)dt$ by a proper substitution $x = \phi(t)$, where $\int g(t)dt$ can be obtained by using standard forms or some known method. Now, let us prove the theorem which is called the rule of substitution for integration.

Theorem 6.4 : $g : [\alpha, \beta] \to R$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). $g'(t)$ is continuous on (α, β) and $g'(t) \neq 0$, $\forall t \in (\alpha, \beta)$. $R_g \subset [a, b]$ and $f : [a, b] \to R$ is continuous, then $x = g(t)$, gives

$$\int f(x)dx = \int f(g(t))\ g'(t)dt.$$

Proof : Since f is continuous on $[a, b]$, $\int f(x)dx$ exists. Now, $x = g(t)$ is continuous on $[\alpha, \beta]$ and $f(x)$ is continuous on $[a, b]$.

So $f(g(t))$ is also continuous on $[\alpha, \beta]$ and $g'(t)$ is given to be continuous. Hence $f(g(t))\ g'(t)$ is continuous. So,

$\int f(g(t)) \cdot g'(t)dt$ also exists.

Let $h(x) = \int f(x)dx$

$\therefore \quad h'(x) = f(x)$

Since $x = g(t)$

$\therefore \quad h'(g(t)) = f(g(t))$

As h is a differentiable function of x and x is a differentiable function of t, h is a differentiable function of t.

$$\therefore \quad \frac{d}{dt} h(g(t)) = \frac{d}{dt}((hog)(t))$$
$$= h'(g(t))\ g'(t)$$
$$= f(g(t))\ g'(t)$$

$$\therefore \quad \frac{d}{dt} h(g(t)) = f(g(t))\ g'(t)$$

$$\therefore \quad h(g(t)) = \int f(g(t))\ g'(t)dt$$

$$\therefore \quad h(x) = \int f(g(t))\ g'(t)dt$$

$$\therefore \quad \int f(x)dx = \int f(g(t))\ g'(t)dt$$

Here on the left hand side, we have a function of x. On the right hand side, we have a function of t. Since $g'(t)$ is continuous and non-zero, $x = g(t)$ is one-one function. Hence $t = g^{-1}(x)$ can convert the function on the right hand side into a function of x.

In this rule, a new variable is introduced replacing the variable x. Hence, it is called the method of change of variable also.

Note : (1) In the formula for the method of substitution, $g(t) = x$ converts the right hand side according to $\int f(x)dx = \int f(x) \frac{dx}{dt} dt$.

(2) According to the definition, for $y = f(x)$, $f'(x) = \frac{dy}{dx}$.

Here, $\frac{dy}{dx}$ is not ratio of dy and dx.

But $f'(x) = \frac{(dy)}{(dx)}$ where dx and dy are 'differentials' of x and y respectively. Thus, we can write $dy = f'(x)dx$. Hence, if $t = sinx$, then $dt = cosx\ dx$. (We will study this in the next semester.)

(3) Commonly used functions e^x, $sinx$, $cosx$, $secx$ satisfy the conditions of the theorem on some interval. Thus we will not verify these conditions every time.

Theorem 6.5 : If $\int f(x)dx = F(x)$, then $\int f(ax+b)dx = \frac{1}{a}F(ax+b)$ where $f : I \to R$ is continuous on some interval I. $(a \neq 0)$.

Proof : Let $t = ax + b$. So $x = \frac{t-b}{a}$.

Hence, $x = g(t)$ is continuous and differentiable and $\frac{dx}{dt} = g'(t) = \frac{1}{a} \neq 0$. Also $g'(t)$ is continuous.

$$\therefore \quad \int f(ax+b)dx = \int f(t)\frac{dx}{dt}dt$$

$$= \int f(t)\frac{1}{a}dt$$

$$= \frac{1}{a}\int f(t)dt$$

$$= \frac{1}{a}F(t)$$

$$= \frac{1}{a}F(ax+b)$$

Thus, (1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ gives $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

(2) $\int \frac{1}{x}dx = \log|x| + c$ gives $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b| + c$

(3) $\int cosx\, dx = sinx + c$ gives $\int cos(ax+b)dx = \frac{1}{a}sin(ax+b) + c$

(4) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ gives $\int \frac{1}{(px+q)^2-(a)^2}dx = \frac{1}{p}\cdot\frac{1}{2a}\log\left|\frac{(px+q)-(a)}{(px+q)+(a)}\right| + c$

We can also use all standard forms stated earlier in this manner.

Theorem 6.6 : $[f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$, $(n \neq -1, f(x) > 0)$ where f, f' are continuous and $f'(x) \neq 0$.

Proof : Let $t = f(x)$. So $1 = f'(x)\frac{dx}{dt}$

Again $f'(x) \neq 0$ and is continuous implies $t = f(x)$ is one-one and

$$\int [f(x)]^n f'(x)dx = \int [f(x)]^n \left(f'(x)\frac{dx}{dt}\right)dt$$

$$= \int t^n \cdot 1\, dt$$

$$= \frac{t^{n+1}}{n+1} + c$$

$$\therefore \quad \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c \qquad (t = f(x))$$

Thus, (1) $\int sin^2x\, cosx\, dx = \int (sinx)^2\left(\frac{d}{dx}sinx\right)dx = \frac{(sinx)^{2+1}}{2+1} + c = \frac{sin^3x}{3} + c$

(2) $\int \frac{\sqrt{tanx}}{cos^2x}dx = \int (tanx)^{\frac{1}{2}} sec^2x\, dx = \int (tanx)^{\frac{1}{2}}\left(\frac{d}{dx}tanx\right)dx = \frac{(tanx)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$$= \frac{2}{3}(tanx)^{\frac{3}{2}} + c$$

(3) $\int \frac{x}{\sqrt{x^2+5}}\, dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+5}}\, dx$

$$= \frac{1}{2}\int (x^2+5)^{-\frac{1}{2}} \frac{d}{dx}(x^2+5)dx = \frac{1}{2}\,\frac{(x^2+5)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \sqrt{x^2+5} + c$$

Theorem 6.7 : If f is continuous in $[a, b]$ and differentiable in (a, b) and f' is continuous and non-zero, $\forall x \in [a, b]$ and $f(x) \neq 0$, $\forall x \in [a, b]$, then $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$.

Proof : f' is continuous and non-zero. Hence, f is monotonic (increasing or decreasing) function. Substitution $t = f(x)$ gives $x = f^{-1}(t)$

$\therefore\ f'(x)\frac{dx}{dt} = 1$

Now, $\int \frac{f'(x)}{f(x)} dx = \int \frac{f'(x)}{f(x)} \cdot \frac{dx}{dt}\, dt$

$= \int \frac{1}{t}\, dt$

$= \log|t| + c$

$\therefore\ \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

Thus,

(1) $\int \frac{x}{x^2-15}\, dx = \frac{1}{2}\int \frac{2x}{x^2-15}\, dx$

$$= \frac{1}{2}\int \frac{\frac{d}{dx}(x^2-15)}{x^2-15}\, dx = \frac{1}{2}\log|x^2-15| + c$$

(2) $\int \frac{2cosx - 3sinx}{6cosx + 4sinx}\, dx = \frac{1}{2}\int \frac{-6sinx + 4cosx}{6cosx + 4sinx}\, dx$

$$= \frac{1}{2}\int \frac{\frac{d}{dx}(6cosx + 4sinx)}{(6cosx + 4sinx)}\, dx$$

$$= \frac{1}{2}\log|6cosx + 4sinx| + c$$

6.6 Some More Standard Forms

(16) On any interval I = $\left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$\int tanx\, dx = \log|secx| + c$.

Here, $\int tanx\ dx = \int \frac{secx\ tanx}{secx}\, dx$ **($secx \neq 0$)**

On given interval, $t = secx$ is continuous and differentiable and non-zero and $\frac{dt}{dx} = secx\ tanx$ is also continuous and non-zero.

Taking, $t = secx$, $dt = secx\ tanx\ dx$

$\therefore\ \int tanx\ dx = \int \frac{secx\ tanx}{secx}\, dx$

$= \int \frac{1}{t}\, dt$

$= \log |t| + c$

$= \log |secx| + c$

(17) On any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$\int cotx\ dx = \log |sinx| + c.$

Here, $\int cotx\ \ dx = \int \frac{cosx}{sinx}\ dx$

On given interval, $t = sinx$ is continuous and differentiable and non-zero and $\frac{dt}{dx} = cosx$ is also continuous and non-zero.

Taking $t = sinx$, $dt = cosx\ dx$

$\therefore\ \int cotx\ dx = \int \frac{cosx}{sinx}\ dx$

$= \int \frac{1}{t}\ dt$

$= \log |t| + c$

$= \log |sinx| + c$

(18) On any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$\int cosecx\ dx = \log |cosecx - cotx| + c$, $x \neq k\pi$, $k \in Z$

$= \log |tan\frac{x}{2}| + c$

On given interval, $1 - cosx \neq 0$ and $sinx \neq 0$

$\therefore\ cosecx - cotx = \frac{1 - cosx}{sinx} \neq 0$ in the domain.

Now, $I = \int cosecx\ dx = \int \frac{cosecx\ (cosecx - cotx)}{(cosecx - cotx)}\ dx$

$= \int \frac{cosec^2x - cosecx\ cotx}{cosecx - cotx}\ dx$

Now, $t = cosecx - cotx$ is continuous and differentiable and non-zero and $\frac{dt}{dx} = cosec^2x - cosecx\ cotx$ is continuous and non-zero on given interval.

$\therefore\ I = \int \frac{1}{t}\ dt$

$= \log |t| + c$

$= \log |cosecx - cotx| + c$

Again, $\log |cosecx - cotx| = \log \left|\frac{1 - cosx}{sinx}\right|$

$= \log \left|\frac{2sin^2\frac{x}{2}}{2sin\frac{x}{2}cos\frac{x}{2}}\right|$

$= \log \left|tan\frac{x}{2}\right|$

Thus, $\int cosecx\ dx = \log |cosecx - cotx| + c$

$= \log \left|tan\frac{x}{2}\right| + c$

(19) On any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

$\int secx\ dx = \log|secx + tanx| + c$

$= \log\left|tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$

$secx + tanx = \frac{1+sinx}{cosx}$ is defined and non-zero as $x \neq (4k-1)\frac{\pi}{2}$, $k \in Z$

On given interval, $1 + sinx \neq 0$ and $cosx \neq 0$

Now, $I = \int secx\ dx = \int \frac{secx\ (secx + tanx)}{secx + tanx}\ dx$

Now, $t = secx + tanx$ is continuous and differentiable and non-zero and $\frac{dt}{dx} = secx\ tanx + sec^2x$ is continuous and non-zero on given interval.

$\therefore\ I = \int \frac{sec^2x + secx\,tanx}{secx + tanx}\ dt$

$= \int \frac{1}{t}\ dt$

$= \log|t| + c$

$= \log|secx + tanx| + c$

Again, $\log|secx + tanx| = \log\left|\frac{1+sinx}{cosx}\right|$

$= \log\left|\frac{sin^2\frac{x}{2} + cos^2\frac{x}{2} + 2sin\frac{x}{2}cos\frac{x}{2}}{cos^2\frac{x}{2} - sin^2\frac{x}{2}}\right|$

$= \log\left|\frac{(cos\frac{x}{2} + sin\frac{x}{2})^2}{(cos^2\frac{x}{2} - sinx^2\frac{x}{2})}\right|$

$= \log\left|\frac{cos\frac{x}{2} + sin\frac{x}{2}}{cos\frac{x}{2} - sin\frac{x}{2}}\right|$

$= \log\left|\frac{1 + tan\frac{x}{2}}{1 - tan\frac{x}{2}}\right|$

$= \log\left|tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$

Thus, $\int secx\ dx = \log|secx + tanx| + c$

$= \log\left|tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$

Example 6 : Evaluate : $\int \frac{2x^3 + 5x^2 + 3x + 1}{2x - 1}\ dx$

Solution : $I = \int \frac{2x^3 + 5x^2 + 3x + 1}{2x - 1}\ dx$

$= \int \frac{(2x-1)(x^2 + 3x + 3) + 4}{2x - 1}\ dx$

$$= \int \left(x^2 + 3x + 3 + \frac{4}{2x-1}\right) dx$$

$$= \int x^2 dx + 3 \int x \, dx + 3 \int dx + 4 \int \frac{1}{2x-1} \, dx$$

$$= \frac{x^3}{3} + 3 \frac{x^2}{2} + 3x + 4 \times \frac{1}{2} \log | 2x - 1 | + c$$

$$= \frac{x^3}{3} + \frac{3}{2}x^2 + 3x + 2 \log | 2x - 1 | + c$$

Example 7 : Evaluate : $\int \left(\frac{1}{\sqrt{16-9x^2}} + \frac{1}{25-9x^2}\right) dx$

Solution : $I = \int \left(\frac{1}{\sqrt{16-9x^2}} + \frac{1}{25-9x^2}\right) dx$

$$= \int \frac{1}{\sqrt{(4)^2 - (3x)^2}} dx + \int \frac{1}{(5)^2 - (3x)^2} \, dx$$

$$= \frac{1}{3} \sin^{-1}\left(\frac{3x}{4}\right) + \frac{1}{2(5)} \times \frac{1}{3} \log \left|\frac{5+3x}{5-3x}\right| + c$$

$$= \frac{1}{3} \sin^{-1} \frac{3x}{4} + \frac{1}{30} \log \left|\frac{5+3x}{5-3x}\right| + c$$

Example 8 : Evaluate : $\int (7x + 5)\sqrt{3x+2} \, dx$

Solution : We will find m and n such that

$7x + 5 = m(3x + 2) + n$

$7x + 5 = 3mx + 2m + n$

Comparing the coefficient of x and constant term on both sides,

$3m = 7$ and $2m + n = 5$

$\therefore$ $m = \frac{7}{3}$ and $\frac{14}{3} + n = 5$. Thus $n = 5 - \frac{14}{3} = \frac{1}{3}$

$\therefore$ $I = \int [m(3x + 2) + n]\sqrt{3x+2} \, dx$

$$= \int \left[\frac{7}{3}(3x + 2) + \frac{1}{3}\right]\sqrt{3x+2} \, dx$$

$$= \int \left[\frac{7}{3}(3x + 2)^{\frac{3}{2}} + \frac{1}{3}(3x + 2)^{\frac{1}{2}}\right] dx$$

$$= \frac{7}{3} \int (3x + 2)^{\frac{3}{2}} dx + \frac{1}{3} \int (3x + 2)^{\frac{1}{2}} \, dx$$

$$= \frac{7}{3} \frac{(3x+2)^{\frac{5}{2}}}{3 \times \frac{5}{2}} + \frac{1}{3} \frac{(3x+2)^{\frac{3}{2}}}{3 \times \frac{3}{2}} + c$$

$$= \frac{14}{45} (3x + 2)^{\frac{5}{2}} + \frac{2}{27} (3x + 2)^{\frac{3}{2}} + c$$

Example 9 : Evaluate : $\int \frac{3x+4}{\sqrt{4x+5}}\,dx$

Solution : $I = \int \frac{3x+4}{\sqrt{4x+5}}\,dx$

$$= \int \frac{\frac{3}{4}(4x+5)+\frac{1}{4}}{\sqrt{4x+5}}\,dx$$

$$= \frac{3}{4}\int \frac{4x+5}{\sqrt{4x+5}}\,dx + \frac{1}{4}\int \frac{1}{\sqrt{4x+5}}\,dx$$

$$= \frac{3}{4}\int (4x+5)^{\frac{1}{2}}\,dx + \frac{1}{4}\int (4x+5)^{-\frac{1}{2}}\,dx$$

$$= \frac{3}{4}\,\frac{(4x+5)^{\frac{3}{2}}}{4\times\frac{3}{2}} + \frac{1}{4}\,\frac{(4x+5)^{\frac{1}{2}}}{4\times\frac{1}{2}} + c$$

$$= \frac{1}{8}(4x+5)^{\frac{3}{2}} + \frac{1}{8}(4x+5)^{\frac{1}{2}} + c$$

Example 10 : Evaluate $\int sin^4x\ cos^4x\ dx.$

Solution : $I = \int sin^4x\ cos^4x\ dx.$

$$= \frac{1}{16}\int (2sinx\ cosx)^4\,dx$$

$$= \frac{1}{16}\int (sin2x)^4\,dx$$

$$= \frac{1}{16}\int \left(\frac{1-cos4x}{2}\right)^2 dx$$

$$= \frac{1}{64}\int (1-2cos\,4x+cos^2 4x)\,dx$$

$$= \frac{1}{64}\int \left(1-2cos\,4x+\left(\frac{1+cos\,8x}{2}\right)\right) dx$$

$$= \frac{1}{128}\int (3-4cos\,4x+cos\,8x)\,dx$$

$$= \frac{1}{128}\left[3x-\frac{4sin4x}{4}+\frac{sin8x}{8}\right]+c$$

$$= \frac{1}{128}\left[3x-sin\,4x+\frac{1}{8}sin\,8x\right]+c$$

Example 11 : Evaluate : $\int sinax\ cosbx\ dx,\ a \neq \pm b$

Solution : $I = \int (sinax\ cosbx)\,dx$

$$= \frac{1}{2}\int (2sinax\ cosbx)\,dx$$

$$= \frac{1}{2}\int [sin\,(ax+bx)+sin(ax-bx)]\,dx$$

$$= \frac{1}{2}\int [sin\,(a+b)x\,dx + \frac{1}{2}\int sin(a-b)x\,dx$$

$$= -\frac{1}{2}\,\frac{cos\,(a+b)x}{a+b} - \frac{1}{2}\,\frac{cos\,(a-b)x}{a-b} + c$$

$$= -\frac{1}{2}\left[\frac{cos\,(a+b)x}{a+b}+\frac{cos\,(a-b)x}{a-b}\right]+c$$

Example 12 : Evaluate : $\int sinx\ sin2x\ sin3x\ dx$

Solution : $I = \int sinx\ sin2x\ sin3x\ dx$

$= \frac{1}{2}\int (2sin\ 2x \cdot sinx)\ sin3x\ dx$

$= \frac{1}{2}\int (cosx - cos3x)\ sin3x\ dx$

$= \frac{1}{4}\int (2sin\ 3x\ cosx - 2sin\ 3x\ cos3x)\ dx$

$= \frac{1}{4}\int (sin\ 4x + sin\ 2x - sin\ 6x)\ dx$

$= \frac{1}{4}\left[-\frac{cos4x}{4} - \frac{cos2x}{2} + \frac{cos6x}{6}\right] + c$

$= \frac{1}{24}\ cos6x - \frac{1}{16}\ cos4x - \frac{1}{8}\ cos2x + c$

Example 13 : Evaluate : $\int \frac{1}{sin\,(x-a)\ cos\,(x-b)}\ dx$

Solution : $I = \int \frac{1}{sin\,(x-a)\ cos\,(x-b)}\ dx$

$= \frac{1}{cos\,(a-b)} \int \frac{cos\,(a-b)}{sin\,(x-a)\ cos\,(x-b)}\ dx$

$= \frac{1}{cos\,(a-b)} \int \frac{cos\,[(x-a)-(x-b)]}{sin\,(x-a)\ cos\,(x-b)}\ dx$ $\quad (cos(b-a) = cos(a-b))$

$= \frac{1}{cos\,(a-b)} \int \frac{cos(x-a)\ cos(x-b) + sin\,(x-a)\ sin\,(x-b)}{sin(x-a) \cdot cos(x-b)}\ dx$

$= \frac{1}{cos\,(a-b)} \int [cot(x-a) + tan(x-b)]\ dx$

$= \frac{1}{cos\,(a-b)}\ [\log|\,sin(x-a)\,| - \log|\,cos(x-b)\,|] + c$

$= \frac{1}{cos\,(a-b)}\ \log\left|\frac{sin(x-a)}{cos(x-b)}\right| + c$

Example 14 : Evaluate : $\int \frac{sinx\ cosx}{3sin^2x - 4cos^2x}\ dx$

Solution : $I = \int \frac{sinx\ cosx}{3sin^2x - 4cos^2x}\ dx$

Let $3sin^2x - 4cos^2x = t$

$\therefore$ $[3(2sinx\ cosx) + 4(2cosx\ sinx)]dx = dt$

$\therefore$ $14sinx\ cosx\ dx = dt$

$\therefore$ $sinx\ cosx\ dx = \frac{1}{14}\ dt$

$\therefore$ $I = \frac{1}{14}\int \frac{1}{t}\ dt$

$= \frac{1}{14}\ \log|\,t\,| + c$

$= \frac{1}{14}\ \log|\,3sin^2x - 4sin^2x\,| + c$

Example 15 : Evalaute $\int \frac{1}{2-3\cos 2x}\,dx$

Solution : $I = \int \frac{1}{2-3\cos 2x}\,dx$

$$= \int \frac{1}{2-3\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)}\,dx$$

$$= \int \frac{\sec^2 x\,dx}{2(1+\tan^2 x)-3+3\tan^2 x}$$

$$= \int \frac{\sec^2 x\,dx}{5\tan^2 x-1}$$

Taking $\tan x = t$, $\sec^2 x\,dx = dt$

$$\therefore \quad I = \int \frac{dt}{5t^2-1}$$

$$= \int \frac{dt}{(\sqrt{5}t)^2-(1)^2}$$

$$= \frac{1}{2\sqrt{5}} \log\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right| + c$$

$$= \frac{1}{2\sqrt{5}} \log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right| + c$$

Example 16 : Evaluate : $\int \frac{\cos x}{\sqrt[3]{1-9\sin x}}\,dx \quad (\sin x < \frac{1}{9})$

Solution : $I = \int \frac{\cos x}{\sqrt[3]{1-9\sin x}}\,dx$

Taking $1 - 9\sin x = t^3$, $-9\cos x\,dx = 3t^2\,dt$

$\therefore \quad \cos x\,dx = -\frac{1}{3}t^2\,dt$

$$\therefore \quad I = \int \frac{-\frac{1}{3}t^2\,dt}{\sqrt[3]{t^3}}$$

$$= -\frac{1}{3}\int t\,dt$$

$$= -\frac{1}{3}\left(\frac{t^2}{2}\right) + c$$

$$= -\frac{1}{6}(1-9\sin x)^{\frac{2}{3}} + c$$

Example 17 : Evaluate $\int \frac{\cos^9 x}{\sin x}\,dx$

Solution : $I = \int \frac{\cos^9 x}{\sin x}\,dx$

Taking $\sin x = t$, $\cos x\,dx = dt$

$$\therefore \quad I = \int \frac{(\cos^2 x)^4}{\sin x}\cos x\,dx$$

$$= \int \frac{(1-\sin^2 x)^4}{\sin x}\cos x\,dx$$

$$= \int \frac{(1-t^2)^4}{t}\,dt$$

$$= \int \frac{1-4t^2+6t^4-4t^6+t^8}{t}\,dt$$

$$= \int \left(\frac{1}{t} - 4t + 6t^3 - 4t^5 + t^7\right) dt$$

$$= \log|t| - 4\frac{t^2}{2} + \frac{6t^4}{4} - \frac{4t^6}{6} + \frac{t^8}{8} + c$$

$$= \log|\sin x| - 2\sin^2 x + \frac{3}{2}\sin^4 x - \frac{2}{3}\sin^6 x + \frac{1}{8}\sin^8 x + c$$

Example 18 : Evaluate $\int \frac{x^2 \sin^{-1}(x^3)}{\sqrt{1-x^6}}\,dx$

Solution : $I = \int \frac{x^2 \sin^{-1}(x^3)}{\sqrt{1-x^6}}\,dx$

Taking $\sin^{-1}x^3 = t$, $\frac{3x^2dx}{\sqrt{1-x^6}} = dt$

i.e. $\frac{x^2\,dx}{\sqrt{1-x^6}} = \frac{1}{3}dt$

$$\therefore \quad I = \int \sin^{-1}(x^3) \cdot \frac{x^2\,dx}{\sqrt{1-x^6}}$$

$$= \int \frac{1}{3}t \cdot dt$$

$$= \frac{1}{3}\left[\frac{t^2}{2}\right] + c$$

$$= \frac{1}{6}[\sin^{-1}(x^3)]^2 + c$$

Exercise 6.2

Integrate the following functions defined on proper domain *w.r.t.* *x*.

1. $\frac{1}{5x-3}$

2. $e^{7x+4} + (5x-3)^8$

3. $\frac{7^{2x+3}\sin^2 2x + \cos^2 2x}{\sin^2 2x}$

4. $5^{4x+3} - 3\sin(2x+3)$

5. $\frac{1}{\sqrt{5x^2-4}}$

6. $\frac{1}{\sqrt{16-9x^2}}$

7. $\frac{1}{\sqrt{5x^2+3}} + \frac{1}{9-4x^2}$

8. $\frac{1}{\sqrt{2x^2+3}} + \frac{1}{7x^2+3}$

9. $\frac{(2x+1)^2}{x-2}$

10. $\frac{x^5+2}{x+1}$

11. $\frac{1}{\sqrt{5-3x}}$

12. $3^{5x-2} + \frac{1}{(2x+1)^3}$

13. $\cot^2(3+5x)$

14. $\sin^2(3x+5)$

15. $\frac{1-\cos 3x}{\sin^2 3x}$

16. $\sqrt{1+\cos x}$, $0 < x < \pi$

17. $\frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}$

18. $\frac{1}{\sqrt{5-2x} + \sqrt{3-2x}}$

19. $\frac{x+2}{(x+1)^2}$

20. $\frac{x^2+1}{(x+1)^2}$

21. $\frac{x^3+3x^2+2x+1}{x-1}$

22. $x\sqrt{x+3}$

23. $\frac{x}{\sqrt{x+1}}$

24. $\frac{x+1}{\sqrt{2x+1}}$

25. $\frac{8x+13}{\sqrt{4x+7}}$

26. $\cos^4 x$

27. $\sin^3 x \cos^3 x$

28. $\sin^3(2x-1)$

29. $\cos 2x \cdot \cos 4x$

30. $\frac{\sin 4x}{\sin x}$

31. $\cos 2x \cdot \cos 4x \cdot \cos 6x$

32. $\frac{1}{\sqrt{1-\cos x}}$

33. $\sqrt{\frac{1+\cos x}{1-\cos x}}$ $0 < x < \pi$

34. $\sin mx \cdot \sin nx,\ m \neq n,\ m, n \in \mathbb{N}$

35. $\frac{\sin x}{\sin(x-a)}$

36. $\frac{1}{\sin(x-a)\sin(x-b)}$

37. $\frac{3x+2}{3-2x}$

38. $(3x^2-4x+5)^{\frac{3}{2}}(3x-2)$

39. $\frac{x+3}{\sqrt{x^2+6x+4}}$

40. $x^3\sqrt{5x^4+3}$

41. $\frac{\sin^2(\log x)}{x}$

42. $\frac{\sqrt{1+\log x}}{x}$

43. $\frac{\sin 2x}{(m+n\cos 2x)^2}$

44. $\frac{1-\tan x}{1+\tan x}$

45. $\frac{e^x(1+x)}{\cos^2(xe^x)}$

46. $e^{-x}\operatorname{cosec}^2(2e^{-x}+3)$

47. $\frac{x^{e-1}+e^{x-1}}{x^e+e^x}$

48. $\frac{(3\tan^2 x+2)\sec^2 x}{(\tan^3 x+2\tan x+9)^2}$

49. $\frac{\sin 2x}{(b\cos^2 x + a\sin^2 x)^2}$

50. $\frac{\tan x}{a^2+b^2\tan^2 x}$, $(a < b)$

51. $\frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}$

52. $\frac{(\tan^{-1}x)^{-\frac{3}{2}}}{1+x^2}$

53. $\dfrac{e^x \log(\sin e^x)}{\tan e^x}$

54. $\dfrac{\log(x+1) - \log x}{x(x+1)}$

55. $\tan^3 x$

56. $\sec^4 x \ \tan x$

57. $\tan^6 x$

58. $\dfrac{1}{\sqrt{x+1} + \sqrt[3]{x+1}}$

59. $\dfrac{x^2}{(x+2)^{\frac{1}{3}}}$

60. $\dfrac{1}{a^2\cos^2 x + b^2\sin^2 x}$

61. $\dfrac{1}{3 - 2\sin^2 x}$

62. $\dfrac{\sin x}{\sin 3x}$

63. $\dfrac{1}{8\cos^2 x + 3\sin^2 x + 1}$

64. $\dfrac{1}{3\sin^2 x + \cos 2x}$

*

6.7 Trigonometric Substitutions

Sometimes using proper trigonometric substitutions, we can transform given integrand into a form whose integration can be easily obtained. Particularly, when expressions like $x^2 - a^2$, $a^2 - x^2$, $x^2 + a^2$ occur under square root in integrand, trigonometric substitutions are very useful.

Suppose our aim is to obtain $\int \frac{x^2}{\sqrt{4-x^2}}\, dx$, $(x > 0)$

Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\, d\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$

$$\therefore \quad I = \int \frac{x^2}{\sqrt{4-x^2}}\, dx$$

$$= \int \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}} \cdot 2\cos\theta\, d\theta$$

$$= \int \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{2\cos\theta} \qquad \left(\cos\theta > 0 \text{ as } \theta \in \left(0, \frac{\pi}{2}\right)\right)$$

$$= 4\int \sin^2\theta\, d\theta$$

$$= 4\int \frac{1 - \cos 2\theta}{2}\, d\theta$$

$$= 2\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

$$= 2\theta - 2\sin\theta\cos\theta + c$$

Now, $x = 2\sin\theta$. Hence $\theta = \sin^{-1}\left(\frac{x}{2}\right)$, $\theta \in \left(0, \frac{\pi}{2}\right)$

$$2\sin\theta\cos\theta = 2 \cdot \frac{x}{2}\sqrt{1 - \frac{x^2}{4}} = \frac{1}{2}x\sqrt{4 - x^2}$$

$$\therefore \quad I = 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + c$$

Following is a list of some frequently used substitutions. Mostly they are used to remove radical sign from the integrand. Usually we will take $0 < \theta < \frac{\pi}{2}$.

Integrands	Substitution
$\sqrt{x^2+a^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
$\sqrt{2ax-x^2}$	$x = 2a\sin^2\theta$
$\sqrt{2ax-x^2} = \sqrt{a^2-(x-a)^2}$	$x - a = a\sin\theta$ or $a\cos\theta$

Example 19 : Evaluate : $\int \frac{1}{x\sqrt{x^4-b^4}}\,dx$

Solution : Here, $I = \int \frac{1}{x\sqrt{x^4-b^4}}\,dx$

Let $x^2 = b^2\sec\theta$ $\quad (0 < \theta < \frac{\pi}{2})$

$\therefore \quad 2x\,dx = b^2\sec\theta\ \tan\theta\ d\theta$

Now, $I = \int \frac{2x\,dx}{2x^2\sqrt{x^4-b^4}}$

$= \int \frac{b^2\sec\theta\tan\theta\,d\theta}{2b^2\sec\theta\sqrt{b^4\sec^2\theta-b^4}}$

$= \frac{1}{2b^2}\int d\theta$

$= \frac{1}{2b^2}(\theta) + c$

But, since $x^2 = b^2\sec\theta$, $\sec\theta = \frac{x^2}{b^2}$, $\theta = \sec^{-1}\frac{x^2}{b^2}$ $\quad (0 < \theta < \frac{\pi}{2})$

$\therefore \quad I = \frac{1}{2b^2}\sec^{-1}\left(\frac{x^2}{b^2}\right) + c$

Example 20 : Evaluate : $\int \frac{\sqrt{3-x}}{x}\,dx$, $0 < x < 3$

Solution : Here, $I = \int \frac{\sqrt{3-x}}{x}\,dx$

Let $x = 3\sin^2\theta$ $\quad (0 < \theta < \frac{\pi}{2})$

Then $dx = 3(2\sin\theta\ \cos\theta)\ d\theta$

$$\therefore \quad I = \int \frac{\sqrt{3-3\sin^2\theta}}{3\sin^2\theta}\, 6\sin\theta\,\cos\theta\, d\theta$$

$$= \int \frac{2\sqrt{3}\cos^2\theta}{\sin\theta}\, d\theta$$

$$= 2\sqrt{3} \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$$

$$= 2\sqrt{3} \int (\operatorname{cosec}\theta - \sin\theta)\, d\theta$$

$$= 2\sqrt{3}\, [\log |\operatorname{cosec}\theta - \cot\theta| + \cos\theta] + c$$

But, since $\sin^2\theta = \frac{x}{3}$, $\cos^2\theta = 1 - \frac{x}{3}$. So $\cos\theta = \sqrt{\frac{3-x}{3}}$

$\operatorname{cosec}^2\theta = \frac{3}{x}$. So $\operatorname{cosec}\theta = \sqrt{\frac{3}{x}}$

Also $1 + \cot^2\theta = \frac{3}{x}$. So $\cot\theta = \sqrt{\frac{3}{x} - 1} = \sqrt{\frac{3-x}{x}}$

$$\therefore \quad I = 2\sqrt{3}\left[\log\left|\left(\sqrt{\frac{3}{x}} - \sqrt{\frac{3-x}{x}}\right)\right| + \sqrt{\frac{3-x}{3}}\right] + c$$

Example 21 : Evaluate $\int \frac{\sqrt{x^2+1}}{x^4}\, dx$, $(x < 0)$

Solution : $I = \int \frac{\sqrt{x^2+1}}{x^4}\, dx$

Let $\theta = \tan^{-1}x$, $-\frac{\pi}{2} < \theta < 0$. So, $x = \tan\theta$.

$\therefore \quad dx = \sec^2\theta\, d\theta, \quad \theta \in \left(-\frac{\pi}{2}, 0\right)$

$$\therefore \quad I = \int \frac{\sqrt{\tan^2\theta + 1}}{\tan^4\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int \frac{\sec\theta \cdot \sec^2\theta}{\tan^4\theta}\, d\theta \qquad \left(\sec\theta > 0 \text{ as } \theta \in \left(-\frac{\pi}{2}, 0\right)\right)$$

$$= \int \frac{\cos\theta}{\sin^4\theta}\, d\theta$$

$$= \int (\sin\theta)^{-4} \frac{d}{d\theta}(\sin\theta)\, d\theta$$

$$= \frac{(\sin\theta)^{-3}}{-3} + c$$

$$= -\frac{1}{3}\, \frac{1}{\sin^3\theta} + c$$

$$= -\frac{1}{3} \operatorname{cosec}^3\theta + c$$

Now, $\tan\theta = x$. So $\cot\theta = \frac{1}{x}$

and $\operatorname{cosec}\theta = -\sqrt{1 + \cot^2\theta} = -\sqrt{1 + \frac{1}{x^2}} = \frac{-\sqrt{x^2+1}}{|x|} = \frac{-\sqrt{x^2+1}}{-x} = \frac{\sqrt{x^2+1}}{x}$ $\quad \left(-\frac{\pi}{2} < \theta < 0\right)$

$\therefore \quad I = -\frac{1}{3}\left(\frac{\sqrt{x^2+1}}{x}\right)^3 + c$

$= -\frac{1}{3}\frac{(1+x^2)^{\frac{3}{2}}}{x^3} + c$

Example 22 : Evaluate : $\int \frac{1}{(x-1)^{\frac{3}{2}}(x-2)^{\frac{1}{2}}}\,dx,\ (x > 2)$

Solution : $I = \int \frac{dx}{(x-1)^{\frac{3}{2}}(x-2)^{\frac{1}{2}}}$

Let $x - 1 = sec^2\theta$, $dx = 2sec\theta\ sec\theta\ tan\theta\ d\theta$ $\quad (0 < \theta < \frac{\pi}{2})$

$\therefore \quad dx = 2\ sec^2\theta\ tan\theta\ d\theta$

$\therefore \quad I = \int \frac{2sec^2\theta\, tan\theta\, d\theta}{(sec^2\theta)^{\frac{3}{2}}(sec^2\theta - 1)^{\frac{1}{2}}}$

$= \int \frac{2sec^2\theta\, tan\theta\, d\theta}{sec^3\theta \cdot tan\theta}$

$= 2\int cos\theta\ d\theta$

$= 2sin\theta + c$

Now, $sec^2\theta = x - 1$. So $cos^2\theta = \frac{1}{x-1}$

and $sin^2\theta = 1 - cos^2\theta = 1 - \frac{1}{x-1} = \frac{x-2}{x-1}$

$\therefore \quad sin\theta = \sqrt{\frac{x-2}{x-1}}$ $\quad \left(0 < \theta < \frac{\pi}{2}\right)$

$\therefore \quad I = 2\sqrt{\frac{x-2}{x-1}} + c$

6.8 An Important Substitution

If the integrand is $\frac{1}{a+bsinx}$, $\frac{1}{a+bcosx}$ or $\frac{1}{a+bsinx+c\,cosx}$, then $tan\frac{x}{2} = t$ is a useful substitution. Using this substitution, we can transform integrand into a standard form of t.

Taking $tan\frac{x}{2} = t$, $sec^2\frac{x}{2} \cdot \frac{1}{2}\,dx = dt$

$\therefore \quad dx = \frac{2dt}{sec^2\frac{x}{2}} = \frac{2dt}{1+tan^2\frac{x}{2}} = \frac{2dt}{1+t^2}$

$sinx = \frac{2tan\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$ and $cosx = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$

This will transform the integrand into a function of t.

Example 23 : Evaluate : $\int \frac{1}{1-2\sin x}\, dx$

Solution : Let $\tan\frac{x}{2} = t$. So, $dx = \frac{2dt}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$

$$\therefore \quad I = \int \frac{1}{1-2\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int \frac{1}{t^2-4t+1}\, dt$$

$$= 2\int \frac{1}{t^2-4t+4-3}\, dt$$

$$= 2\int \frac{1}{(t-2)^2-(\sqrt{3})^2}\, dt$$

$$= 2 \times \frac{1}{2\sqrt{3}} \log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}} \log\left|\frac{\tan\frac{x}{2}-2-\sqrt{3}}{\tan\frac{x}{2}-2+\sqrt{3}}\right| + c$$

Example 24 : Evaulate $\int \frac{dx}{\cos\alpha+\cos x}$, $\quad \alpha \in \left(0, \frac{\pi}{2}\right)$

Solution : $I = \int \frac{dx}{\cos\alpha+\cos x}$

Let $\tan\frac{x}{2} = t$. So $dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore \quad I = \int \frac{1}{\cos\alpha+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{2\,dt}{\cos\alpha+t^2\cdot\cos\alpha+1-t^2}$$

$$= 2\int \frac{dt}{(1+\cos\alpha)-(1-\cos\alpha)t^2}$$

$$= 2\int \frac{dt}{2\cos^2\frac{\alpha}{2}-2\sin^2\frac{\alpha}{2}\cdot t^2}$$

$$= \int \frac{dt}{\left(\cos\frac{\alpha}{2}\right)^2-\left(t\sin\frac{\alpha}{2}\right)^2}$$

$$= \frac{1}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} \log\left|\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\,t}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\,t}\right| + c$$

$$= \frac{1}{\sin\alpha} \log\left|\frac{1+\tan\frac{\alpha}{2}\cdot\tan\frac{x}{2}}{1-\tan\frac{\alpha}{2}\cdot\tan\frac{x}{2}}\right| + c$$

6.9 Integrals of the type $\int sin^m x \ cos^n x \ dx$ $m, n \in N$

If $m, n \in N$, the following cases may occur :

(1) m, n are odd

(2) m is odd and n is even.

(3) m is even and n is odd

(4) m and n both are even.

Let $I = \int sin^m x \ cos^n x \ dx$

Case 1 : m, n are odd.

We may take $sinx = t$ or $cosx = t$. Usually if $m > n$, $sinx = t$ and if $n > m$, $cosx = t$ will be convenient.

Case 2 : m is odd and n is even.

We take $cosx = t$.

Case 3 : m is even and n is odd.

We take $sinx = t$.

Case 4 : m and n both are even.

In this situation, we transform $sin^m x \ cos^n x$ using $sin^2 x = \frac{1 - cos2x}{2}$ and $cos^2 x = \frac{1 + cos2x}{2}$

For small values of m and n, these methods are simple. For larger and negative values of m and n, other methods are availables, but at this stage we will not study them.

Example 25 : Evaluate $\int cos^2 x \ sin^5 x \ dx$

Solution : Here, $m = 5$ is odd. $n = 2$ is even.

$\therefore$ Let $cosx = t$. So $-sinx \ dx = dt$

$\therefore$ $sinx \ dx = -dt$

$$I = \int cos^2 x \ sin^5 x \ dx$$

$$= \int sin^4 x \cdot cos^2 x \cdot sinx \ dx$$

$$= \int (1 - cos^2 x)^2 \cdot cos^2 x \ sinx \ dx$$

$$= \int (1 - t^2)^2 \ t^2 \ (-dt)$$

$$= \int (1 - 2t^2 + t^4)(-t^2) \ dt$$

$$= \int (2t^4 - t^6 - t^2) \ dt$$

$$= \frac{2t^5}{5} - \frac{t^7}{7} - \frac{t^3}{3} + c$$

$$= \frac{2}{5}cos^5 x - \frac{1}{7}cos^7 x - \frac{1}{3}cos^3 x + c$$

Example 26 : Evaluate $\int sin^{23} x \cdot cos^3 x \ dx$

Solution : $I = \int sin^{23} x \cdot cos^3 x \ dx$

Here, $m = 23$, $n = 3$. m and n both are odd.

But $m > n$. Let $sinx = t$, so $cosx \ dx = dt$

$$I = \int sin^{23} x \ cos^2 x \ cosx \ dx$$

$$= \int sin^{23} x \ (1 - sin^2 x) \ cosx \ dx$$

$$= \int t^{23} \ (1 - t^2) \ dt$$

$$= \int (t^{23} - t^{25}) \ dt$$

$$= \frac{t^{24}}{24} - \frac{t^{26}}{26} + c$$

$$= \frac{sin^{24} x}{24} - \frac{sin^{26} x}{26} + c$$

Example 27 : Evaluate $\int sin^2 x \ cos^4 x \ dx$

Solution : $I = \int sin^2 x \ cos^4 x \ dx$

Here, m and n both are even.

$$\therefore \quad \sin^2 x\cos^4 x = \frac{1}{4}(4\sin^2 x\cos^2 x)\ \cos^2 x$$

$$= \frac{1}{4}\sin^2 2x \cdot \cos^2 x$$

$$= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{16}(1 - \cos 4x + \cos 2x - \cos 4x\ \cos 2x)$$

$$= \frac{1}{16}\left[1 - \cos 4x + \cos 2x - \frac{(2\cos 4x\cos 2x)}{2}\right]$$

$$= \frac{1}{32}[2 - 2\cos 4x + 2\cos 2x - \cos 6x - \cos 2x]$$

$$= \frac{1}{32}(2 - 2\cos 4x + \cos 2x - \cos 6x)$$

$$\therefore \quad I = \frac{1}{32}\int [2 + \cos 2x - 2\cos 4x - \cos 6x]\ dx$$

$$= \frac{1}{32}\left[2x + \frac{\sin 2x}{2} - \frac{2\sin 4x}{4} - \frac{\sin 6x}{6}\right] + c$$

$$= \frac{1}{192}[12x + 3\sin 2x - 3\sin 4x - \sin 6x] + c$$

Exercise 6.3

Integrate the following functions defined on proper domains using trigonometric substition :

1. $\dfrac{1}{x^2\sqrt{1-x^2}}$ $(|x| < 1)$
2. $\dfrac{\sqrt{9-x^2}}{x^2}$, $(0 < x < 3)$
3. $\dfrac{1}{(a^2+x^2)^{\frac{3}{2}}}$
4. $x^2\sqrt{a^6-x^6}$, $(0 < x < a)$
5. $\dfrac{1}{\sqrt{2ax-x^2}}$ $(0 < x < 2a)$
6. $\sqrt{\dfrac{2-x}{x}}$ $(0 < x < 2)$
7. $\sqrt{\dfrac{a-x}{a+x}}$ $(0 < x < a)$
8. $\dfrac{x^2}{\sqrt{a^6-x^6}}$ $(0 < x < a)$
9. $\dfrac{1}{x^2(1+x^2)^2}$
10. $\dfrac{x}{(16-9x^2)^{\frac{3}{2}}}$ $\left(0 < x < \frac{4}{3}\right)$
11. $\dfrac{x^2}{(x^2-a^2)^{\frac{3}{2}}}$ $(|x| > |a|)$
12. $x\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ $(0 < x < a)$
13. $\dfrac{1}{(x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}}$ $(x > 2)$
14. $\dfrac{\sqrt{25-x^2}}{x^2}$ $(0 < x < 5)$
15. $\dfrac{1}{1+\sin x+\cos x}$
16. $\dfrac{1}{3+2\sin x+\cos x}$
17. $\dfrac{1}{5+4\cos x}$
18. $\dfrac{1}{1+\cos\alpha\cos x}$
19. $\dfrac{1}{2-\cos x}$
20. $\dfrac{1}{\cos x-\sin x}$ $\left(0 < x < \frac{\pi}{4}\right)$

21. $\sin^4 x \cos^3 x$

22. $\sin^3 x \cos^{10} x$

23. $\cos^3 x \sin^7 x$

24. $\sin^5 x \cos^4 x$

25. $\sin^5 x$

26. $\sin^4 x \cos^2 x$

*

6.10 Integration of the type (1) $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

(2) $\int \frac{Ax+B}{ax^2+bx+c}\,dx$ and $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$

(1) To evaluate this type of integrals, we express $ax^2 + bx + c$ as the sum or difference of two squares.

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$$

$$= a\,[(x + \alpha)^2 - \beta^2], \text{ if } b^2 - 4ac > 0, \text{ where } \beta^2 = \frac{b^2-4ac}{4a^2}$$

$$= a\,[(x + \alpha)^2 + \beta^2], \text{ if } b^2 - 4ac < 0, \text{ where } \beta^2 = -\frac{b^2-4ac}{4a^2}$$

Thus, $ax^2 + bx + c = a\,[(x + \alpha)^2 \pm \beta^2]$. Hence $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ can be evaluated using previous standard forms. Now let us understand the method by the following examples :

(**Note :** If $b^2 = 4ac$, then $ax^2 + bx + c = a\left(x+\frac{b}{2a}\right)^2$)

Example 28 : Evaluate : $\int \frac{dx}{3x^2+13x-10}$

Solution : $I = \int \frac{dx}{3x^2+13x-10}$

$$= \frac{1}{3}\int \frac{dx}{x^2+\frac{13}{3}x-\frac{10}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{x^2+\frac{13}{3}x+\left(\frac{13}{6}\right)^2-\left(\frac{13}{6}\right)^2-\frac{10}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}$$

$$= \frac{1}{3} \times \frac{1}{2\left(\frac{17}{6}\right)} \log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right| + c$$

$$= \frac{1}{17} \log \left| \frac{x - \frac{2}{3}}{x + 5} \right| + c$$

$$= \frac{1}{17} \log \left| \frac{3x - 2}{3(x + 5)} \right| + c$$

$$\Bigg(\text{Note : } I = \frac{1}{17} \log \left| \frac{3x - 2}{3(x + 5)} \right| + c$$

$$= \frac{1}{17} \Big[\log |3x - 2| - \log 3 - \log |x + 5|\Big] + c$$

$$= \frac{1}{17} \log \left| \frac{3x - 2}{x + 5} \right| + c' \text{ where } c' = c - \frac{1}{17} \log 3\Bigg)$$

Example 29 : Evaluate : $\int \frac{1}{\sqrt{x(1 - 2x)}} \, dx$ $\quad \left(0 < x < \frac{1}{2}\right)$

Solution : $I = \int \frac{1}{\sqrt{x(1 - 2x)}} \, dx$

$$= \int \frac{1}{\sqrt{x - 2x^2}} \, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{x}{2} - x^2}} \, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{1}{16} - \left(x^2 - \frac{x}{2} + \frac{1}{16}\right)}} \, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}} \, dx$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x - \frac{1}{4}}{\frac{1}{4}} \right) + c$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} (4x - 1) + c$$

(2) In order to evaluate this type of integrals, first we find constants m and n such that,

$Ax + B = m(\text{derivative of } ax^2 + bx + c) + n$

$Ax + B = m(2ax + b) + n$

$Ax + B = (2ma)x + (mb + n)$

Comparing the coefficient of x and constant term on both sides, we get

$A = 2ma$ and $mb + n = B$

$\therefore \quad m = \frac{A}{2a}$ and $n = B - mb$

Now, $\int \frac{Ax + B}{ax^2 + bx + c} \, dx = \int \frac{m(2ax+b) + n}{ax^2 + bx + c} \, dx$

$$= m \int \frac{2ax+b}{ax^2 + bx + c} \, dx + n \int \frac{1}{ax^2 + bx + c} \, dx$$

$$= m \log | ax^2 + bx + c | + n \int \frac{1}{ax^2 + bx + c} \, dx$$

For the first integral, we use $\int \frac{f'(x)}{f(x)} \, dx = \log |f(x)|$ and for the second integral, we have to use method (1) of making perfect square in the denominator.

Now, for $\int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} \, dx = \int \frac{m(2ax + b) + n}{\sqrt{ax^2 + bx + c}} \, dx$

$$= \int \frac{m(2ax + b)}{\sqrt{ax^2 + bx + c}} \, dx + n \int \frac{1}{\sqrt{ax^2 + bx + c}} \, dx$$

$$= m \int (ax^2 + bx + c)^{-\frac{1}{2}} (2ax + b)dx + n \int \frac{1}{\sqrt{ax^2 + bx + c}} \, dx$$

$$= m \frac{(ax^2 + bx + c)^{\frac{1}{2}}}{\frac{1}{2}} + n \int \frac{1}{\sqrt{ax^2 + bx + c}} \, dx$$

$$= 2m (ax^2 + bx + c)^{\frac{1}{2}} + n \int \frac{1}{\sqrt{ax^2 + bx + c}} \, dx$$

For the first integral, we use $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$ and to evaluate the second integral, we have to use method (1) of making perfect square in the denominator.

Example 30 : Evaluate : $\int \frac{2x + 3}{3x^2 + 4x + 5} \, dx$

Solution : First, we will find constants m and n such that $2x + 3 = m \frac{d}{dx}(3x^2 + 4x + 5) + n$

$2x + 3 = m(6x + 4) + n$

$2x + 3 = (6m)x + 4m + n$

Comparing coefficient of x and constant term on both sides, we get $6m = 2$ and $4m + n = 3$.

$\therefore$ $m = \frac{1}{3}$ and $\frac{4}{3} + n = 3$. Thus, $n = \frac{5}{3}$

$\therefore$ $I = \int \frac{2x + 3}{3x^2 + 4x + 5} \, dx = \int \frac{\frac{1}{3}(6x + 4) + \frac{5}{3}}{3x^2 + 4x + 5} \, dx$

$$= \frac{1}{3} \int \frac{6x + 4}{3x^2 + 4x + 5} \, dx + \frac{5}{3} \int \frac{1}{3x^2 + 4x + 5} \, dx$$

$$= \frac{1}{3}\int \frac{6x+4}{3x^2+4x+5}\,dx + 5\int \frac{1}{9x^2+12x+4+11}\,dx$$

$$= \frac{1}{3}\int \frac{6x+4}{3x^2+4x+5}\,dx + 5\int \frac{1}{(3x+2)^2+(\sqrt{11})^2}\,dx$$

$$= \frac{1}{3}\log|3x^2+4x+5| + \frac{5}{3\sqrt{11}}\tan^{-1}\frac{3x+2}{\sqrt{11}} + c$$

Example 31 : Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

Solution : Here, the derivative of denominator $x^2 + 4x + 1$ is $2x + 4$. Thus $2x + 3$ in the numerator can be written as $2x + 3 = (2x + 4) - 1$.

$$\therefore \quad I = \int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int \frac{(2x+4)-(1)}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int \frac{(2x+4)}{\sqrt{x^2+4x+1}}\,dx - \int \frac{1}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int (x^2+4x+1)^{-\frac{1}{2}}(2x+4)\,dx - \int \frac{1}{\sqrt{(x+2)^2-(\sqrt{3})^2}}\,dx$$

$$= \frac{(x^2+4x+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \log|(x+2)+\sqrt{(x+2)^2-(\sqrt{3})^2}| + c$$

$$= 2\sqrt{x^2+4x+1} - \log|x+2+\sqrt{x^2+4x+1}| + c$$

Example 32 : Evaluate $\int \frac{x^2}{x^4+1}\,dx$

Solution : $I = \int \frac{x^2}{x^4+1}\,dx$

$$= \frac{1}{2}\int \frac{2x^2}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{(x^2+1)+(x^2-1)}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}\,dx + \frac{1}{2}\int \frac{x^2-1}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}\,dx + \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}\,dx$$

$$= \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2} + \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2}$$

Let $x - \frac{1}{x} = u$ for the first integral and $x + \frac{1}{x} = v$ for the second integral.

So $\left(1+\frac{1}{x^2}\right) dx = du$ and $\left(1-\frac{1}{x^2}\right) dx = dv$

$$\therefore \quad I = \frac{1}{2} \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{dv}{v^2 - (\sqrt{2})^2}$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{1}{2} \times \frac{1}{2\sqrt{2}} \log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1} \frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}} + \frac{1}{4\sqrt{2}} \log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{4\sqrt{2}} \log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + c$$

Exercise 6.4

Integrate the following *w.r.t.* *x*.

1. $\frac{1}{x^2+3x+3}$
2. $\frac{1}{4x^2-4x+3}$
3. $\frac{1}{1-6x-9x^2}$
4. $\frac{1}{3+2x-x^2}$
5. $\frac{1}{\sqrt{x^2-x+5}}$
6. $\frac{1}{\sqrt{2x^2+3x-2}}$
7. $\frac{1}{\sqrt{7-3x-2x^2}}$
8. $\frac{1}{\sqrt{3x^2+5x+7}}$
9. $\frac{1}{\sqrt{(x-1)(x-2)}}$
10. $\frac{1}{\sqrt{9+8x-x^2}}$
11. $\frac{4x+1}{x^2+3x+2}$
12. $\frac{3x+2}{2x^2+x+1}$
13. $\frac{2x+3}{\sqrt{x^2+4x+5}}$
14. $\frac{3x+1}{\sqrt{5-2x-x^2}}$
15. $\frac{2\sin 2x - \cos x}{6-\cos^2 x - 4\sin x}$
16. $\frac{e^x}{\sqrt{5-4e^x-e^{2x}}}$
17. $\frac{x^2}{\sqrt{x^6+2x^3+3}}$
18. $\frac{2x}{\sqrt{1-x^2-x^4}}$
19. $\frac{x^2+1}{x^4+1}$
20. $\frac{x^2+4}{x^4+16}$
21. $\frac{x^2+1}{x^4+7x^2+1}$
22. $\frac{1}{x^4+1}$
23. $\frac{x^2-1}{x^4+x^2+1}$
24. $\frac{x^2}{x^4+x^2+1}$

*

Miscellaneous Examples

Example 33 : Evaluate : $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)} dx$

Solution : $I = \int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)}\,dx$

$$= \int \frac{\sin 2x}{\sin^2 x - \sin^2\frac{\pi}{3}}\,dx$$

$$= \int \frac{\sin 2x}{\sin^2 x - \frac{3}{4}}\,dx$$

$$= \int \frac{\frac{d}{dx}\left(\sin^2 x - \frac{3}{4}\right)}{\sin^2 x - \frac{3}{4}}\,dx$$

$$= \log\left|\sin^2 x - \frac{3}{4}\right| + c$$

Example 34 : Evaluate : $\int \frac{1}{x(x^n+1)}\,dx \qquad (x > 0)$

Solution : $I = \int \frac{1}{x(x^n+1)}\,dx$

Let $x^n + 1 = t$. Then $nx^{n-1}\,dx = dt$

$$\therefore \quad I = \int \frac{nx^{n-1}\,dx}{nx^n\,(x^n+1)}$$

$$= \frac{1}{n}\int \frac{dt}{(t-1)t}$$

$$= \frac{1}{n}\int \frac{1}{t^2 - t + \frac{1}{4} - \frac{1}{4}}\,dt$$

$$= \frac{1}{n}\int \frac{1}{\left(t-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\,dt$$

$$= \frac{1}{n}\log\left|\frac{\left(t-\frac{1}{2}\right)-\left(\frac{1}{2}\right)}{\left(t-\frac{1}{2}\right)+\left(\frac{1}{2}\right)}\right| + c$$

$$= \frac{1}{n}\log\left|\frac{t-1}{t}\right| + c$$

$$= \frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right) + c$$

Second Method :

$$I = \frac{1}{n}\int \frac{dt}{(t-1)t}$$

$$= \frac{1}{n}\int \frac{[t-(t-1)]\,dt}{(t-1)t}$$

$$= \frac{1}{n}\left[\int \frac{dt}{t-1} - \int \frac{dt}{t}\right]$$

$$= \frac{1}{n}\left[\log|t-1| - \log|t|\right] + c$$

$$= \frac{1}{n}\log\left|\frac{t-1}{t}\right| + c$$

$$= \frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right) + c$$

Example 35 : Evaluate : $\int \sqrt{\frac{\sin(x-\theta)}{\sin(x+\theta)}}\,dx \qquad \theta < x < \frac{\pi}{2} + \theta,\ 0 < x < \frac{\pi}{2}$

Solution : $I = \int \sqrt{\frac{\sin(x-\theta)}{\sin(x+\theta)}}\,dx$

$$= \int \sqrt{\frac{\sin(x-\theta)}{\sin(x+\theta)} \times \frac{\sin(x-\theta)}{\sin(x-\theta)}}\,dx \qquad (\sin(x-\theta) > 0)$$

$$= \int \frac{\sin(x-\theta)}{\sqrt{\sin^2 x - \sin^2\theta}}\,dx$$

$$= \int \frac{\sin x \cos\theta - \cos x \sin\theta}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx - \sin\theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x}{\sqrt{1 - \cos^2 x - 1 + \cos^2\theta}}\, dx - \sin\theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\theta}}\, dx$$

$$= \cos\theta \int \frac{\sin x\, dx}{\sqrt{\cos^2\theta - \cos^2 x}} - \sin\theta \int \frac{\cos x\, dx}{\sqrt{\sin^2 x - \sin^2\theta}}$$

Let $\cos x = u$ in the first integral and $\sin x = v$ in the second integral.

$\therefore \quad -\sin x\, dx = du$ and $\cos x\, dx = dv$

$$\therefore \quad I = \cos\theta \int \frac{-du}{\sqrt{\cos^2\theta - u^2}} - \sin\theta \int \frac{dv}{\sqrt{v^2 - \sin^2\theta}}$$

$$= -\cos\theta \sin^{-1}\left(\frac{u}{\cos\theta}\right) - \sin\theta \log \left| v + \sqrt{v^2 - \sin^2\theta} \right| + c$$

$$= -\cos\theta \sin^{-1}\left(\frac{\cos x}{\cos\theta}\right) - \sin\theta \log \left| \sin x + \sqrt{\sin^2 x - \sin^2\theta} \right| + c$$

Example 36 : Evaluate : $\int \frac{\sin x}{\sqrt{1 + \sin x}}\, dx \quad 0 < x < \pi$

Solution : $I = \int \frac{(\sin x + 1) - 1}{\sqrt{1 + \sin x}}\, dx$

$$= \int \sqrt{1 + \sin x}\, dx - \int \frac{1}{\sqrt{1 + \sin x}}\, dx$$

$$= \int \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx - \int \frac{1}{\sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}}\, dx$$

$$= \int \left|\sin\frac{x}{2} + \cos\frac{x}{2}\right| dx - \int \frac{1}{\left|\sin\frac{x}{2} + \cos\frac{x}{2}\right|}\, dx$$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx - \int \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\frac{x}{2} + \frac{1}{\sqrt{2}}\sin\frac{x}{2}\right)}\, dx \qquad \left(0 < \frac{x}{2} < \frac{\pi}{2}\right)$$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx - \int \frac{1}{\sqrt{2}\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)}\, dx$$

$$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) dx - \frac{1}{\sqrt{2}} \int \sec\left(\frac{x}{2} - \frac{\pi}{4}\right) dx$$

$$= \frac{-\cos\frac{x}{2}}{\frac{1}{2}} + \frac{\sin\frac{x}{2}}{\frac{1}{2}} - \frac{1}{\sqrt{2}} \times \frac{1}{\left(\frac{1}{2}\right)} \log\left|\sec\left(\frac{x}{2} - \frac{\pi}{4}\right) + \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)\right| + c$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \sqrt{2} \log\left|\sec\left(\frac{x}{2} - \frac{\pi}{4}\right) + \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)\right| + c$$

Exercise 6

Integrate the following with respect to x.

1. $\dfrac{\sqrt{x}}{\sqrt{x}+\sqrt[3]{x}}$ $(x > 0)$

2. $\dfrac{\log\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}$

3. $\dfrac{1}{(x+1)^2\sqrt{x^2+2x+2}}$

4. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$, $x \in (0, 1)$

5. $\sqrt{\dfrac{x+3}{x+2}}$ $(x > -2)$

6. $\dfrac{x^2+5x+3}{x^2+3x+2}$ $(x \neq -2, -1)$

7. $\dfrac{x^2}{x^2+7x+10}$ $(x \neq -5, -2)$

8. $\dfrac{1}{\cos(x-a)\cos(x-b)}$

9. $\dfrac{\sin(x+a)}{\sin(x+b)}$

10. $x(1-x)^n$

11. $\sqrt{\tan x}$

12. $\dfrac{1}{\sin^4 x+\cos^4 x}$

13. $\dfrac{1}{1-2a\cos x+a^2}$, $0 < a < 1$

14. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

Section A

(1) If $\int f(x)dx = \dfrac{(\log x)^5}{5} + c$, then $f(x) =$ ...... ☐

(a) $\dfrac{\log x}{4}$ (b) $\dfrac{(\log x)^5}{5}$ (c) $\dfrac{(\log x)^4}{x}$ (d) $\dfrac{(\log x)^6}{6}$

(2) $\int e^{x \log a} e^x \, dx =$ ...... $+ c$ ☐

(a) $a^x \cdot e^x$ (b) $\dfrac{(ae)^x}{(1+\log a)}$ (c) $\dfrac{e^x}{\log(ae)}$ (d) $\dfrac{a^x}{1+\log_e a}$

(3) $\int \dfrac{(\log x)^3}{x} dx =$ ...... $+ c$ ☐

(a) $(\log x)^2$ (b) $\dfrac{(\log x)^2}{2}$ (c) $\frac{1}{4}(\log x)^4$ (d) $\frac{2}{3}(\log x)^3$

(4) $\int \sec^2\left(5-\frac{x}{2}\right) dx =$ ...... $+ c$ ☐

(a) $\tan\left(5-\frac{x}{2}\right)$ (b) $2\tan\left(5-\frac{x}{2}\right)$ (c) $-2\tan\left(5-\frac{x}{2}\right)$ (d) $-\frac{1}{2}\tan\left(5-\frac{x}{2}\right)$

(5) $\int \dfrac{1}{4x^2+9} dx =$ ...... $+ c$ ☐

(a) $\frac{1}{3}\tan^{-1}\left(\frac{2x}{3}\right)$ (b) $\frac{1}{4}\tan^{-1}\left(\frac{2x}{3}\right)$ (c) $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)$ (d) $\frac{3}{2}\tan^{-1}\left(\frac{2x}{3}\right)$

(6) $\int \sqrt{1-\cos x}\ dx = \ldots\ldots + c,\ 2\pi < x < 3\pi$

(a) $-2\sqrt{2}\ \cos\frac{x}{2}$ (b) $-\sqrt{2}\ \cos\frac{x}{2}$ (c) $2\sqrt{2}\ \cos\frac{x}{2}$ (d) $-\frac{1}{2}\cos\left(\frac{x}{2}\right)$

(7) $\int \frac{dx}{x\sqrt{3+\log x}} = \ldots\ldots + c$

(a) $2\sqrt{3+\log x}$ (b) $\frac{2}{\sqrt{3+\log x}}$ (c) $\sqrt{3+\log x}$ (d) $-2\sqrt{3+\log x}$

(8) $\int \frac{1}{\sqrt{4-3x}}\ dx = \ldots\ldots + c$

(a) $-\frac{2}{3}(4-3x)^{-\frac{1}{2}} + c$ (b) $-\frac{2}{3}(4+3x)^{\frac{1}{2}}$

(c) $-\frac{2}{3}(4-3x)^{\frac{1}{2}}$ (d) $\frac{2}{3}(4+3x)^{\frac{1}{2}}$

(9) $\int \frac{x-2}{x^2-4x+5}\ dx = \ldots\ldots + c$

(a) $\log|x^2-4x+5| + c$ (b) $\log\sqrt{x^2-4x+5}$

(c) $\frac{1}{2}(x^2-4x+5)^2$ (d) $\log\left(\frac{x-3}{x-1}\right)$

(10) $\int \frac{1}{3t^2+4}\ dt = \ldots\ldots + c$

(a) $\frac{1}{12}\tan^{-1}\left(\frac{3t}{4}\right)$ (b) $\frac{1}{3}\log\left|\frac{t+2}{t-2}\right|$

(c) $\frac{1}{2\sqrt{3}}\ \tan^{-1}\left(\frac{\sqrt{3}t}{2}\right)$ (d) $\frac{1}{2\sqrt{3}}\ \tan^{-1}\left(\frac{3t}{4}\right)$

(11) $\int \frac{1}{1-\cos t}\ dt = \ldots\ldots + c$

(a) $\operatorname{cosec} t + \cot t$ (b) $-\cot\frac{t}{2}$ (c) $-4\cot\frac{t}{2}$ (d) $\operatorname{cosec} t + \cot t$

(12) $\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}\ dx = \ldots\ldots + c$

(a) $e \cdot 3^{-3x}$ (b) $e^3 \log x$ (c) $\frac{x^3}{3}$ (d) $\frac{x^2}{3}$

(13) $\int \sec^2 x \cdot \operatorname{cosec}^2 x\ dx = \ldots\ldots + c$

(a) $\tan x + \cot x$ (b) $\tan x - \cot x$ (c) $\sec^2 x + \operatorname{cosec}^2 x$ (d) $\cot x - \tan x$

(14) $\int e^{3\log x} \cdot (x^4+1)^{-1}\ dx = \ldots\ldots + c$

(a) $\log(x^4+1)$ (b) $-\log(x^4+1)$ (c) $\frac{1}{4}\log(x^4+1)$ (d) $\frac{-3}{(x^4+1)^2}$

(15) $\int \frac{(\log x)^4}{x}\ dx = \ldots\ldots + c$

(a) $\frac{(\log x)^5}{5}$ (b) $\frac{(\log x)^2}{2}$ (c) $\frac{\log x^5}{5x}$ (d) $\log x \cdot (\log x)^4 + \frac{(\log x)^5}{5x}$

(16) $\int \frac{dx}{\sqrt{1-x}} = ...$

(a) $sin^{-1}\sqrt{x} + c$ (b) $-2\sqrt{1-x} + c$ (c) $-sin^{-1}\sqrt{x} + c$ (d) $2\sqrt{1-x} + c$

(17) $\int \frac{(sinx)^{99}}{(cosx)^{101}} dx = ... + c$

(a) $\frac{(tanx)^{100}}{100}$ (b) $\frac{(tanx)^{2}}{2}$ (c) $\frac{(tanx)^{98}}{98}$ (d) $\frac{(tanx)^{97}}{97}$

(18) $\int \frac{\log x^2}{x} dx = ...$

(a) $\log|x^2| + c$ (b) $\log x + c$ (c) $(\log x)^2 + c$ (d) $\frac{1}{2}(\log x)^2 + c$

(19) $\int \frac{xsinx}{(xcosx - sinx + 5)} dx = ... + c$

(a) $\log | xcosx - sinx + 5 |$ (b) $-\log | xcosx - sinx + 5 |$

(c) $\log | xsinx - cosx + 5 |$ (d) $-\log | xsinx - cosx + 5 |$

(20) $\int (1 - cosx)cosec^2x \, dx = ... + c$

(a) $tan\frac{x}{2}$ (b) $cot\frac{x}{2}$ (c) $\frac{1}{2}tan\frac{x}{2}$ (d) $2\,tan\frac{x}{2}$

Section B

(21) If $f'(x) = x^2 + 5$, then $\int f(x)dx = ...$. (c and k are arbitrary constants)

(a) $\frac{x^4}{12} + \frac{5x^2}{8} + cx + k$ (b) $-\frac{x^4}{12} - \frac{5x^2}{2} - cx + k$

(c) $\frac{x^4}{12} - \frac{5x^2}{12} + cx + k$ (d) $\frac{x^4}{12} + \frac{5x^2}{2} + cx + k$

(22) $\int \frac{10x^9 + 10^x \log 10}{10^x + x^{10}} dx = ... + c$

(a) $10^x - x^{10}$ (b) $10^x + x^{10}$ (c) $(10^x - x^{10})^{-1}$ (d) $\log|10^x + x^{10}|$

(23) $\int cos^3x \cdot e^{\log sinx} dx = ... + c$

(a) $-\frac{sin^4x}{4}$ (b) $\frac{e^{sinx}}{4}$ (c) $\frac{e^{cosx}}{4}$ (d) $\frac{-cos^4x}{4}$

(24) $\int \frac{sinx}{1+4cosx} dx = ... + c$

(a) $\log|1 + 4cosx|$ (b) $-4\log|1 + 4cosx|$

(c) $-\frac{1}{4}\log|1 + 4cosx|$ (d) $-\log|1 + 4cosx|$

(25) $\int \frac{1}{\sqrt{x+3}-\sqrt{x}}\,dx = \ldots\ldots + c$

(a) $\frac{3}{2}(x+2)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}}$ (b) $\frac{2}{9}(x+2)^{\frac{1}{2}} + \frac{2}{9}x^{\frac{1}{2}}$

(c) $\frac{2}{9}(x+3)^{\frac{3}{2}} + \frac{2}{9}x^{\frac{3}{2}}$ (d) $\frac{2}{9}(x+2)^{\frac{3}{2}} + \frac{2}{9}x$

(26) $\int sin2x\,cos\,3x\,dx = A\,cosx + B cos\,5x + c$, then $A + B = \ldots\ldots$

(a) $\frac{1}{5}$ (b) $\frac{3}{10}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$

(27) $\int \frac{cos\,4x+1}{cotx-tanx}\,dx = A\,cos4x + c$, then $A = \ldots\ldots$

(a) $-\frac{1}{2}$ (b) $-\frac{1}{4}$ (c) $-\frac{1}{8}$ (d) $\frac{1}{8}$

(28) $\int \frac{1+cosx}{sinx\,cosx}\,dx = \ldots\ldots + c$

(a) $\log|sinx| + \log|cosx|$ (b) $\log\left|tanx \cdot tan\frac{x}{2}\right|$

(c) $\log\left|1 + tan\frac{x}{2}\right|$ (d) $\log\left|sec\frac{x}{2} + tan\frac{x}{2}\right|$

(29) $\int \frac{sinx-cosx}{sinx+cosx}\,dx = \ldots\ldots + c$

(a) $\frac{1}{sinx+cosx}$ (b) $\frac{1}{sinx-cosx}$

(c) $\log|sinx + cosx|$ (d) $\log\left|\frac{1}{sinx+cosx}\right|$

(30) $\int \frac{1-cosx}{cosx\,(1+cosx)}\,dx = \ldots\ldots + c$

(a) $2\log|cosx| + tan\frac{x}{2}$ (b) $\log|secx + tanx| - 2tan\frac{x}{2}$

(c) $\log|tanx| + 2tan\frac{x}{2}$ (d) $\frac{1}{2}\log|secx| - tan\frac{x}{2}$

(31) $\int \frac{dx}{e^x+e^{-x}} = \ldots\ldots + c$

(a) $\log|e^x - e^{-x}|$ (b) $\log|e^x + e^{-x}|$ (c) $tan^{-1}(e^x)$ (d) $tan^{-1}(e^{2x})$

(32) $\int \frac{dx}{x+x\log x} = \ldots\ldots + c$

(a) $\log|x + x\log x|$ (b) $x\log|1 + \log x|$

(c) $\log|1 + \log x|$ (d) $\frac{1+\log x}{x^2}$

(33) $\int \frac{\sqrt{tanx}}{sinx\,cosx}\,dx = \ldots\ldots + c$

(a) $\frac{\sqrt{tanx}}{2}$ (b) $\frac{\sqrt{cotx}}{2}$ (c) $2\sqrt{cotx}$ (d) $2\sqrt{tanx}$

Section C

(34) $\int \frac{dx}{\cos x - \sin x} = \ldots\ldots + c$ ☐

(a) $\frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{x}{2} + \frac{3\pi}{8}\right) \right|$ (b) $\frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{\pi}{8} + \frac{x}{2}\right) \right|$

(c) $\frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{x}{2} - \frac{3\pi}{8}\right) \right|$ (d) $\log \left| \cos \frac{x}{2} \right|$

(35) $\int \frac{dx}{(1+\sin x)^{\frac{1}{2}}} = \ldots\ldots + c$ ☐

(a) $\sqrt{2} \log \left| \tan\left(\frac{3\pi}{8} - \frac{x}{4}\right) \right|$ (b) $\sqrt{2} \log \left| \operatorname{cosec}\left(\frac{\pi}{8} + \frac{x}{2}\right) - \cot\left(\frac{\pi}{8} + \frac{x}{2}\right) \right|$

(c) $\sqrt{2} \log \left| \tan\left(\frac{\pi}{8} + \frac{x}{4}\right) \right|$ (d) $\sqrt{2} \log \left| \sec\left(\frac{\pi}{2} + \frac{x}{4}\right) + \tan\left(\frac{\pi}{2} + \frac{x}{4}\right) \right|$

(36) $\int \frac{dx}{5 - 4\cos x} = \ldots\ldots + c$ ☐

(a) $\frac{1}{3} \tan^{-1}\left(3 \tan \frac{x}{2}\right)$ (b) $\frac{1}{3} \tan^{-1}\left(\frac{2}{3} \tan \frac{x}{2}\right)$

(c) $\frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right)$ (d) $\frac{2}{3} \tan^{-1}\left(3 \tan \frac{x}{2}\right)$

(37) $\int \frac{\sin x}{\sin(x-a)} dx = \ldots\ldots + c$ ☐

(a) $x \cos a + \sin a \log|\sin(x-a)|$ (b) $(x-a)\cos a - \sin a \log|\sin(x-a)|$

(c) $\sin a \log|\sin(x-a)| + \cos a \, x$ (d) $\sin a \cdot x + \cos a \log|\sin(x-a)|$

(38) $\int \frac{\sin 2x}{p\cos^2 x + q\sin^2 x} dx = \ldots\ldots + c$ ☐

(a) $\frac{q}{p} \log|p \sin 2x + q\cos 2x|$ (b) $(q-p)\log|p\cos^2 x + q\sin^2 x|$

(c) $\frac{1}{q-p} \log|p\cos^2 x + q\sin^2 x|$ (d) $\frac{1}{p^2+q^2} \log|p\cos^2 x + q\sin^2 x|$

(39) $\int \frac{\tan x}{4 + 9\tan^2 x} dx = \ldots\ldots + c$ ☐

(a) $\frac{2}{3} \tan^{-1}\left(\frac{2}{3} \tan x\right)$ (b) $\frac{3}{2} \tan^{-1}\left(\frac{1}{3} \tan x\right)$

(c) $\frac{1}{10} \log|4 + 9\tan^2 x|$ (d) $\frac{1}{10} \log|4\cos^2 x + 9\sin^2 x|$

(40) $\int \sqrt{\frac{a-x}{a+x}} dx = \ldots\ldots + c$ ☐

(a) $\frac{a}{2} \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$ (b) $\frac{1}{a} \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$

(c) $\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$ (d) $a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$

(41) If $\int \frac{\cos^4 x}{\sin^2 x}\,dx = px + q\sin 2x + r\cot x + c$, then

(a) $p = -\frac{3}{2},\ q = -\frac{1}{4},\ r = -1$
(b) $p = -\frac{1}{4},\ q = -\frac{3}{2},\ r = -1$
(c) $p = 1,\ q = -\frac{1}{4},\ r = 1$
(d) $p = \frac{3}{2},\ q = -\frac{1}{4},\ r = 1$

(42) $\int \frac{e^x}{e^{2x} + e^x + 1}\,dx = \ldots\ldots + c$

(a) $\frac{1}{\sqrt{3}} \sec^{-1}\left(\frac{2e^x + 1}{\sqrt{3}}\right)$
(b) $\tan^{-1}(1 + e^x)$
(c) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2e^x + 1}{\sqrt{3}}\right)$
(d) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{e^x + 1}{\sqrt{3}}\right)$

(43) $\int \frac{1}{\sqrt{2 - 3x - x^2}}\,dx = \ldots\ldots + c$

(a) $\sin^{-1}\left(\frac{2 - 3x}{\sqrt{3}}\right)$
(b) $\sin^{-1}\left(\frac{2x - 1}{\sqrt{15}}\right)$
(c) $\sin^{-1}\left(\frac{2x + 3}{\sqrt{17}}\right)$
(d) $\sin^{-1}\left(\frac{3 + 2x}{3\sqrt{2}}\right)$

Section D

(44) $\int \frac{x^2 + 1}{x^4 - x^2 + 1}\,dx = \ldots\ldots + c$

(a) $x \tan^{-1}\left(\frac{x^2 + 1}{x}\right)$
(b) $\tan^{-1}\left(\frac{x^2 - 1}{x}\right)$
(c) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 + 1}{\sqrt{2}x}\right)$
(d) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}x}\right)$

(45) $\int (\sqrt{\tan x} + \sqrt{\cot x})\,dx = \ldots\ldots + c$

(a) $\frac{\tan x}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x + 1}{\sqrt{2\tan x}}\right)$
(b) $\sqrt{2} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right)$
(c) $\sqrt{2} \tan^{-1}\left(\frac{\tan x + 1}{\sqrt{2\tan x}}\right)$
(b) $\frac{\tan x}{\sqrt{2}} \tan^{-1}\left(\frac{\cot x - 1}{\sqrt{2\tan x}}\right)$

(46) $\int \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}}\,\frac{dx}{x} = \ldots\ldots + c$

(a) $2\log\left(\frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1}\right) - 2\sin^{-1}\sqrt{x}$
(b) $\log\left(\frac{\sqrt{1 - x} - 1}{\sqrt{1 + x} + 1}\right) + 2\sin^{-1}\left(\frac{\sqrt{1 - x}}{\sqrt{1 + x}}\right)$
(c) $2\log\left(\frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}}\right) + \frac{1}{2}\cot^{-1}\sqrt{x + 1}$
(b) $\log\left(\frac{\sqrt{1 - x} - 1}{\sqrt{1 - x} + 1}\right) - 2\sin^{-1}\sqrt{x}$

(47) $\int \frac{dx}{(9-x^2)^{\frac{3}{2}}} = \ldots\ldots + c$ ☐

(a) $\frac{x}{3\sqrt{9-x^2}}$ (b) $\frac{x}{9\sqrt{9+x^2}}$ (c) $\frac{x}{9\sqrt{9-x^2}}$ (b) $\frac{x}{(9-x^2)^{\frac{3}{2}}}$

(48) If $\int x^3 \sqrt{\frac{1+x^2}{1-x^2}}\, dx = p\cos^{-1}x^2 + q\sqrt{1-x^4} + rx^2\sqrt{1-x^4} + c$, then $p + q + r = \ldots\ldots$ ☐

(a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (b) -1

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Summary

We have studied the following points in this chapter :

1. Definition of primitive or antiderivative or indefinite integral.
2. Working rules for integration.
3. Standard integrals :

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \in R - \{-1\},\ x \in R^+$.

(2) $\int \frac{1}{x} dx = \log|x| + c,\ x \in R - \{0\}$

(3) $\int \cos x\, dx = \sin x + c,\ \forall x \in R$

(4) $\int \sin x\, dx = -\cos x + c,\ \forall x \in R$

(5) $\int \sec^2 x\, dx = \tan x + c,\ x \neq (2k-1)\frac{\pi}{2},\ k \in Z$

(6) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c,\ x \neq k\pi,\ k \in Z$

(7) $\int \sec x \tan x\, dx = \sec x + c,\ x \neq (2k-1)\frac{\pi}{2},\ k \in Z$

(8) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c,\ x \neq k\pi,\ k \in Z$

(9) $\int a^x dx = \frac{a^x}{\log_e a} + c,\ a \in R^+ - \{1\},\ x \in R$

(10) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c,\quad a \in R - \{0\},\ x \in R$

(11) $\int \frac{dx}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c,\ a \in R - \{0\},\ x \neq \pm a$

(12) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{x+a}{x-a}\right| + c,\ a \in R - \{0\},\ x \neq \pm a$

(13) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c,\ x \in (-a, a),\ a > 0.$

(14) $\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c, \quad |x| > |a| > 0.$

(15) $\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \log|x + \sqrt{x^2 \pm a^2}| + c, \quad \forall x \in R.$

4. Rule of substitution for integration.

5. If $\int f(x)dx = F(x)$, then $\int f(ax+b)dx = \frac{1}{a}F(ax+b)$ where $f : I \to R$ is continuous on some interval I. $(a \neq 0)$.

6. $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$, $(n \neq -1, f(x) > 0)$ where f, f' are continuous and $f'(x) \neq 0$.

7. If f is continuous in $[a, b]$ and differentiable in (a, b) and f' is continuous and non-zero, $\forall x \in [a, b]$ and $f(x) \neq 0$, $\forall x \in [a, b]$, then $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c.$

(16) $\int \tan x\, dx = \log|\sec x| + c,$

on any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

(17) $\int \cot x\, dx = \log|\sin x| + c,$

on any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

(18) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + c, \; x \neq k\pi, \; k \in Z,$

on any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

(19) $\int \sec x\, dx = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c,$

on any interval $I = \left(k\pi, (2k+1)\frac{\pi}{2}\right)$ or $\left((2k-1)\frac{\pi}{2}, k\pi\right)$, $k \in Z$

Classical Period (400 – 1200)

This period is often known as the golden age of Indian Mathematics. This period saw mathematicians such as Aryabhata, Varahamihira, Brahmagupta, Bhaskara I, Mahavira, and Bhaskara II who gave broader and clearer shape to many branches of mathematics. Their contributions would spread to Asia, the Middle East, and eventually to Europe. Unlike Vedic mathematics, their works included both astronomical and mathematical contributions. In fact, mathematics of that period was included in the 'astral science' (jyotisha-shatra) and consisted of three sub-disciplines: mathematical sciences (ganita or tantra), horoscope astrology (hora or jataka) and divination (samhita). This tripartite division is seen in Varahamihira's 6th century compilation—Pancasiddhantika (literally panca, "five," siddhanta, "conclusion of deliberation", dated 575 CE)—of five earlier works, Surya Siddhanta, Romaka Siddhanta, Paulisa Siddhanta, Vasishtha Siddhanta and Paitamaha Siddhanta, which were adaptations of still earlier works of Mesopotamian, Greek, Egyptian, Roman and Indian astronomy. As explained earlier, the main texts were composed in Sanskrit verse, and were followed by prose commentaries.