PROBABILITY 7

The description of right lines and circles upon which geometry is founded belongs to mechanics. Geometry does not teach us to draw these lines but requires them to be drawn.

— Newton

7.1 Introduction

We have already started our study on probability. Recall that the set of all possible outcomes of a random experiment is the sample space and any subset of a sample space is an event. We know the axiomatic definition of probability and related theorems on it. We may also recall the classical definition of probability that if a finite sample space associated with a random experiment has n equally likely outcomes and of these $r\,(0 \le r \le n)$ outcomes are favourable for the occurrence of an event A, then probability of event A namely P(A), is given by $\frac{r}{n}$. Now we elaborate these ideas further.

7.2 Conditional Probability

As we have defined probability, it is meaningless to ask for the probability of an event without referring to a sample space. As an example, if we ask for the probability that an engineer earns at least ₹ 4,00,000 a year is meaningless. We must specify whether we are referring to all engineers in the India, all those in a particular industry, all those in academic field, all those working in a government department and so on. Thus, when we use the symbol P(A) for the probability of an event A, some sample space U must be associated with it. Now we introduce the symbol P(A | B), read as "the probability of A, given B".

The symbol P(A | B) is used to make it clear that the underlying sample space is B. Here, P(A | B) is called the conditional probability of A relative to B. Thus, every probability is a conditional probability. Of course we use the simplified notation P(A) whenever the underlying sample space is U. But whenever the sample space is reduced to some proper subset B, then we write the conditional probability of A, given B as P(A | B). Thus, a conditional probability is the probability of an event given that another event has occurred.

Let us illustrate some of the ideas connected with conditional probabilities. Let us consider the experiment of rolling a pair of balanced dice. We try to find the probability that the total of numbers appearing on the upper face of two dice is greater than 8, given that the number on the first die is 6. Let A be the event that total of the number on top face of two dice is greater than 8 and let B be the event that the first die has 6 on the top face. We wish to find P(A | B). This probability can be computed by considering only those outcomes for which the first die has a 6. Then, determine the favourable outcomes of these. All the possible outcomes for two dice are shown below :

$$U = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

Die 1	Die 2	Total
1	1	2
1	2	3
1	3	4
1	4	5
1	5	6
1	6	7
2	1	3
2	2	4
2	3	5
2	4	6
2	5	7
2	6	8
3	1	4
3	2	5
3	3	6
3	4	7
3	5	8
3	6	9

Die 1	Die 2	Total
4	1	5
4	2	6
4	3	7
4	4	8
4	5	9
4	6	10
5	1	6
5	2	7
5	3	8
5	4	9
5	5	10
5	6	11
6	1	7
6	2	8
6	3	9
6	4	10
6	5	11
6	6	12

Fig. 7.1

There are 6 outcomes for which the first die shows 6, and out of these, there are four outcomes whose total on two dice is more than 8 (6, 3; 6, 4; 6, 5; 6, 6).

$\therefore \quad P(A \mid B) = \frac{4}{6} = \frac{2}{3}$ (i)

Let us look at this example in another way. Note that with respect to the sample space U,

we have $P(A \cap B) = \frac{4}{36}$ $(n = 36, r = 4)$

and $P(B) = \frac{6}{36}$ $(n = 36, r = 6)$

$\therefore \quad \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{36}}{\frac{6}{36}} = \frac{4}{6} = \frac{2}{3}$ (ii)

From (i) and (ii) we see that

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Following these observations, let us now make the following definition :

Conditional Probability : If A and B are any events of S, where S = P(U) and $P(B) \neq 0$, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Let us first prove that the set function $P(A \mid B)$ which is a function of A, is infact a probability function with respect to fixed event B.

Let us recall the axiomatic definition of probability.

Let U be a finite sample space and S be its power set. Suppose that set function $P : S \to R$ satisfies following axioms :

Axiom 1 : $P(A) \geq 0 \quad \forall A \in S$

Axiom 2 : $P(U) = 1$

Axiom 3 : Whenever $A_1, A_2 \in S$ and A_1, A_2 are mutually exclusive,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

Then P is called a probability function on S.

A result : For a fixed event B the set function $P(A|B)$ which is a function of A is a probability function where $P(B) > 0$.

(1) $P(A \cap B) \geq 0$ and $P(B) > 0$.

$\therefore \quad P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq 0$

Hence, for each $A \in S$ and for fixed event B of S, we have $P(A \mid B) \geq 0$. So, conditional probability satisfies the axiom 1 of the probability function.

(2) If A = U, then by the definition of $P(A \mid B)$,

We have $P(U \mid B) = \frac{P(U \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Thus, conditional probability satisfies the axiom 2 of the probability function.

(3) If A_1 and A_2 are mutually exclusive events, then by the definition of conditional probability,

We have $P((A_1 \cup A_2) \mid B) = \frac{P[(A_1 \cup A_2) \cap B]}{P(B)}$ **(i)**

Now, $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$ **(Distributive law)**

Since A_1 and A_2 are mutually exclusive events, $A_1 \cap B$ and $A_2 \cap B$ are also mutually exclusive.

$\therefore \quad P[(A_1 \cup A_2) \cap B] = P(A_1 \cap B) + P(A_2 \cap B)$ **(Axiom 3) (ii)**

$\therefore \quad P((A_1 \cup A_2) \mid B) = \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$ **(by (i) and (ii))**

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 \mid B) + P(A_2 \mid B)$$

So, conditional probability satisfies axiom 3 of the probabilty function.

Thus, conditional probability satisfies all axioms of a probability function.

Properties of Conditional Probability :

(1) If A_1 and A_2 are any two events of a sample space and B is an event of U such that $P(B) \neq 0$, then $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P((A_1 \cap A_2) \mid B)$

We have $P((A_1 \cup A_2) \mid B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$

$$= \frac{P((A_1 \cap B) \cup P(A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B) - P(A_1 \cap A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} - \frac{P(A_1 \cap A_2 \cap B)}{P(B)}$$

$$= P(A_1 \mid B) + P(A_2 \mid B) - P((A_1 \cap A_2) \mid B)$$

(2) $P(A' \mid B) = 1 - P(A \mid B)$, where $P(B) \neq 0$

We have $P(U \mid B) = 1$

$\therefore$ $P((A \cup A') \mid B) = 1$ **($A \cup A' = U$)**

$\therefore$ $P(A \mid B) + P(A' \mid B) = 1$ **(A and A' are disjoint events)**

$\therefore$ $P(A' \mid B) = 1 - P(A \mid B)$

Example 1 : In a box of 100 memory cards of mobile phones, 10 cards have defects of type A, 5 cards have defects of type B and 2 cards have defects of both the types. Find the probabilities that

(1) a card to be drawn at random has a B type defect under the condition that it has an A type defect, and

(2) a card to be drawn at random has no B type defect under the condition that it has no A type defect.

Solution : Let us define the following events :

A : The memory card has A type defect.

B : The memory card has B type defect.

Then by given information

$$P(A) = \frac{10}{100} = 0.10, \; P(B) = \frac{5}{100} = 0.05, \; P(A \cap B) = \frac{2}{100} = 0.02$$

(1) The required probability is given by,

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{0.02}{0.10} = 0.2$$

(2) The required probability is given by

$$P(B' \mid A') = \frac{P(B' \cap A')}{P(A')} = \frac{P((A \cup B)')}{P(A')}$$

$$= \frac{1 - P(A \cup B)}{1 - P(A)}$$

$$= \frac{1 - [P(A) + P(B) - P(A \cap B)]}{1 - P(A)}$$

$$= \frac{1 - (0.10 + 0.05 - 0.02)}{1 - 0.10}$$

$$= \frac{1 - 0.13}{0.90} = \frac{0.87}{0.90} = \frac{87}{90} = \frac{29}{30}$$

Example 2 : The probability that a regularly scheduled flight departs on time is 0.83; the probability that it arrives on time is 0.82; and the probability that it departs and arrives on time is 0.78. Find the probability that a plane (1) arrives on time given that it departed on time, and (2) departed on time given that it has arrived on time.

Solution : Let D be the event that a plane departs on time and A be the event that a plane arrives on time. By the given information we have P(D) = 0.83, P(A) = 0.82, P(D ∩ A) = 0.78

(1) The probability that a plane arrives on time, given that it departed on time is

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = \frac{78}{83}$$

(2) The probability that a plane departed on time given that it has arrived on time is

$$P(D \mid A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = \frac{78}{82} = \frac{39}{41}$$

Example 3 : For a biased die the probabilities for different integers to turn up on top face are given below :

Face	1	2	3	4	5	6
Probability	0.10	0.32	0.21	0.15	0.05	0.17

The die is tossed and 1 or 2 has turned upon top face. What is the probability that it is face 1 ?

Solution : Let A : The event that face 1 turns up

B : The event that face 2 turns up.

From the table, we see that P(A) = 0.10, P(B) = 0.32.

Now, P(A ∪ B) = P(A) + P(B) **(A and B are mutually exclusive events)**

= 0.10 + 0.32 = 0.42

We have to find P(A | (A ∪ B))

$$P(A \mid (A \cup B)) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A \cup B)} \quad \textbf{(Why ?)}$$

$$= \frac{0.10}{0.42} = \frac{10}{42} = \frac{5}{21}$$

Example 4 : A survey of 500 adults inquired about monthly expenses of their child. The survey asked questions about whether or not the person has a child studying in a college and about their monthly expenses. The probabilities are shown in the table below :

	Probability of monthly expenses		
	Expense too much	**Just right**	**Too low**
Child studying in college	0.30	0.13	0.01
Child not studying in college	0.20	0.25	0.11

Suppose a person is chosen at random. Given that the person has a child studying in a college, what is the probability that he or she ranks the expense as "too much" ?

Solution : Let B be the event that randomly chosen person's child studying in a college.

∴ P(B) = 0.30 + 0.13 + 0.01 = 0.44

Let A be the event that randomly chosen person's child's monthly expense is 'too much'.

From the table, we see that P(A $\cap$ B) = P(expense too much $\cap$ child in a college) = 0.30

Hence, required probability $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.30}{0.44} = \frac{15}{22}$

Example 5 : A family has two children. What is the probability that both the children are girls given that at least one of them is a girl ?

Solution : Let b denote a boy and g denote a girl.

The sample space of the experiment is

$U = \{(b, b), (g, b), (b, g), (g, g)\}$.

Let A : The event that both the children are girls.

B : The event that at least one of the child is a girl.

Then $A = \{(g, g)\}$ and $B = \{(b, g), (g, b), (g, g)\}$

$\therefore \quad A \cap B = \{(g, g)\}$

Thus, $P(B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$

$\therefore$ The required probability is $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Exercise 7.1

1. If P(A) = 0.35, P(B) = 0.45 and P(A $\cup$ B) = 0.65, then find P(B | A).
2. If P(A) = 0.40, P(B) = 0.35 and P(A $\cup$ B) = 0.55, then find P(A | B).
3. If P(A) = 0.3, P(B) = 0.5 and P(A | B) = 0.4, then find P(A $\cap$ B) and P(B | A).
4. A balanced die is thrown twice and the sum of the numbers appearing on the top face is observed to be 7. What is the conditional probability that the number 2 has appeared at least once ?
5. A balanced die is rolled. If the outcome is an odd number, what is the probability that it is a prime ?
6. From the table of example 4, find (1) the probability that a person thinks monthly expense of his child is too low given that she is not in a college. (2) The probability that a person thinks monthly expense of his child is just right given that she is in a college.
7. 100 cards numbered 1 to 100 are placed in a box, shuffled thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is a perfect square, what is the probability that it is an odd perfect square ?
8. In a certain town, 40 % residents have computers, 25 % have internet connections and 15 % have both computer and internet connection. A resident is selected at random from the town. (1) If he has a computer, then what is the probability that he has internet connection also ? (2) If he has an internet connection, then determine the probability that he does not have a computer.
9. A balanced die is thrown three times. Let A be the event that 4 appears on the third toss and B be the event that 6 and 5 appears respectively on first two tosses. Find P(A | B).

10. A fair coin is tossed three times. The events A, B, E, F, M, N are described as given (1) A : head on third toss, B : head on first toss. Find $P(A | B)$. (2) E : at least two heads, F : at most two heads. Find $P(E | F)$. (3) M : at the most two tails, N : at least one tail. Find $P(M | N)$.

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7.3 Multiplication Theorem on Probability

We know that the conditional probability of event A given that event B has occured is given by

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

From this result, we can write $P(A \cap B) = P(B) \cdot P(A | B)$ **(i)**

Also, we know that $P(B | A) = \frac{P(B \cap A)}{P(A)}, \ P(A) \neq 0$

$\therefore \quad P(B | A) = \frac{P(A \cap B)}{P(A)}$ **($A \cap B = B \cap A$)**

$\therefore \quad P(A \cap B) = P(A) \cdot P(B | A)$ **(ii)**

Combining (i) and (ii) we get,

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B | A) \quad \text{if } P(A) \neq 0 \\ &= P(B) \cdot P(A | B) \quad \text{if } P(B) \neq 0 \end{aligned}$$

The above result is known as the **Multiplication Rule of Probability.**

Multiplication rule of probability for more than two events : If A, B and C are three events of sample space, we have

$$\begin{aligned} P(A \cap B \cap C) &= P((A \cap B) \cap C) \\ &= P(A \cap B) \ P(C | (A \cap B)) \quad \text{(Multiplication rule of two events)} \\ &= P(A) \ P(B | A) \ P(C | (A \cap B)) \end{aligned}$$

Theorem on total probability :

Theorem 7.1 : If B_1 and B_2 are mutually exclusive and exhaustive events and $P(B_1) \neq 0$, $P(B_2) \neq 0$, then for any event A of S,

$P(A) = P(B_1) \ P(A | B_1) + P(B_2) \ P(A | B_2)$

Proof : Since B_1 and B_2 are mutually exclusive and exhaustive events, we have

$B_1 \cup B_2 = U$ and $B_1 \cap B_2 = \emptyset$

$$\begin{aligned} \therefore \quad A &= A \cap U \\ &= A \cap (B_1 \cup B_2) \\ &= (A \cap B_1) \cup (A \cap B_2) \quad \text{(Distributive law) (i)} \end{aligned}$$

$$\begin{aligned} \text{Now, } (A \cap B_1) \cap (A \cap B_2) &= A \cap (B_1 \cap B_2) \\ &= A \cap \emptyset \quad (B_1 \cap B_2 = \emptyset) \\ &= \emptyset \end{aligned}$$

$\therefore \quad A \cap B_1$ and $A \cap B_2$ are mutually exclusive events

$\therefore \quad$ By (i), $P(A) = P(A \cap B_1) + P(A \cap B_2)$

$P(A) = P(B_1) \ P(A | B_1) + P(B_2) \ P(A | B_2)$ **(Multiplication Rule of Probability)**

Similarly, if B_1, B_2, B_3 are mutually exclusive and exhaustive events and $P(B_1) \neq 0$, $P(B_2) \neq 0$, $P(B_3) \neq 0$, then for any event A of S,

$P(A) = P(B_1) \ P(A | B_1) + P(B_2) \ P(A | B_2) + P(B_3) \ P(A | B_3)$

Bayes' Theorem :

Bayes' theorem is a theorem of probability theory originally stated by the mathematician Reverend Thomas Bayes (1702 - 1761).

Theorem 7.2 : If B_1, B_2 and B_3 are mutually exclusive and exhaustive events and A is any event such that $P(A) \neq 0$, then

$$P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)}, \qquad i = 1, 2, 3$$

Proof : By the definition of conditional probability,

$$P(B_i \mid A) = \frac{P(A \cap B_i)}{P(A)} \qquad \text{(i)}$$

Now, using multiplication rule of probability and theorem on total probability we have

$$P(A \cap B_i) = P(A \mid B_i)\, P(B_i) \qquad \text{(ii)}$$

and $P(A) = P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)$ (iii)

Hence by (i), (ii) and (iii) we get,

$$P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)}, \qquad i = 1, 2, 3$$

$$= \frac{P(A \mid B_i)\, P(B_i)}{\sum_{i=1}^{3} P(A \mid B_i)\, P(B_i)}, \qquad i = 1, 2, 3$$

Independent Events :

If the probability of occurrence or non-occurrence of event B does not affect the probability of occurrence of A i.e. if $P(A \mid B) = P(A)$ then A and B are said to be independent events.

As an example, the event of getting number 6 when a die is rolled first time and the event of getting number 6 when a die is rolled second time are independent events. By contrast, the event of getting number 6 when a die is rolled first time and the event that the sum of the numbers seen on the first and second trials is 8 are not independent.

Now, by the definition of conditional probability,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0)$$

If A and B are independent events, then

$$P(A \mid B) = P(A)$$

$$\therefore \quad \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\therefore \quad P(A \cap B) = P(A) \cdot P(B)$$

Then $P(B) = \frac{P(A \cap B)}{P(A)}$ $(P(A) \neq 0)$

$$\therefore \quad P(B) = P(B \mid A)$$

Thus, **if events A and B are independent and $P(A) > 0$, $P(B) > 0$, then $P(A \cap B) = P(A) \cdot P(B)$ and $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.**

Also, if $P(A \cap B) = P(A) \cdot P(B)$, then we can say that A and B are independent events.

So, if A and B are independent events, $P(A \cap B) = P(A) \cdot P(B)$.

Theorem 7.3 : If A and B are independent events, then A and B', A' and B and A' and B' are also independent.

Proof : Events $A \cap B$ and $A \cap B'$ are mutually exclusive and $A = (A \cap B) \cup (A \cap B')$

$\therefore$ $P(A) = P(A \cap B) + P(A \cap B')$

$\therefore$ $P(A) = P(A) \cdot P(B) + P(A \cap B')$ **(A and B are independent)**

$\therefore$ $P(A \cap B') = P(A)\ (1 - P(B))$

$\therefore$ $P(A \cap B') = P(A)\ P(B')$

$\therefore$ A and B' are independent events. Similarly, we can prove that A' and B are independent events.

Now, $P(A' \cap B') = P[(A \cup B)']$ **(De Morgan's law)**

$= 1 - P(A \cup B)$

$= 1 - (P(A) + P(B) - P(A \cap B))$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A)\ P(B)$ **(A and B are independent)**

$= (1 - P(A)) - P(B)\ (1 - P(A))$

$= (1 - P(A))\ (1 - P(B))$

$P(A' \cap B') = P(A')\ P(B')$

$\therefore$ A' and B' are independent events.

Remark : (1) Three events A, B and C are said to be mutually independent, if

$P(A \cap B) = P(A)\ P(B)$

$P(B \cap C) = P(B)\ P(C)$

$P(A \cap C) = P(A)\ P(C)$

and $P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$

If at least one of the above is not true for three given events, we say that the events are not mutually independent.

(2) Three events A, B and C are said to be pairwise independent, if

$P(A \cap B) = P(A)\ P(B)$

$P(B \cap C) = P(B)\ P(C)$

and $P(A \cap C) = P(A)\ P(C)$

Example 6 : Three cards are drawn in succession, without replacement, from an ordinary deck of 52 playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a ten or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Solution : Here, events A_1 : the first card is a red ace, A_2 : the second card is a ten or a jack, A_3 : the third card is greater than 3 but less than 7.

Now, $P(A_1) = \frac{2}{52}$ **(Ace of heart and diamond)**

$P(A_2 \mid A_1) = \frac{8}{51}$ **(Without replacement, 4 cards of 10 and 4 jacks)**

$P(A_3 \mid (A_1 \cap A_2)) = \frac{12}{50}$ **(Why ?)**

$\therefore$ By multiplication rule of probability,

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | (A_1 \cap A_2))$$

$$= \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{12}{50}$$

$$= \frac{8}{5525}$$

Example 7 : A bag contains 8 red and 5 white balls. Two successive draws of 3 balls are made in such a way that **(1)** balls are replaced before the second trial, **(2)** the balls are not replaced before the second trial. Find the probability that the first draw will give 3 white balls and the second draw will give 3 red balls.

Solution : Let A denote the event of drawing 3 white balls in the first draw and B denote the event of drawing 3 red balls in the second draw. We have to find $P(A \cap B)$.

(1) Draw with replacement : If the balls drawn in the first draw are replaced back in the bag before the 2nd draw, then the events A and B are independent and the required probability is given by the expression :

$$P(A \cap B) = P(A) \cdot P(B)$$

1st draw : 3 balls can be drawn out of $8 + 5 = 13$ balls in $\binom{13}{3}$ ways.

$\therefore$ $n = \binom{13}{3}$

If all the 3 balls drawn are white, then $r = \binom{5}{3}$

$$\therefore \quad P(A) = \frac{r}{n} = \frac{\binom{5}{3}}{\binom{13}{3}}$$

2nd draw : When the balls drawn in the first draw are replaced before the 2nd draw, the bag again contain 13 balls. Now, if all the 3 drawn balls are red, then $r = \binom{8}{3}$

$$\therefore \quad P(B) = \frac{r}{n} = \frac{\binom{8}{3}}{\binom{13}{3}}$$

Hence, $P(A \cap B) = P(A) \cdot P(B)$

$$= \frac{\binom{5}{3}}{\binom{13}{3}} \cdot \frac{\binom{8}{3}}{\binom{13}{3}} = \frac{560}{(286)^2} = \frac{140}{20449}$$

(2) Draw without replacement : If the balls drawn are not replaced back before the second draw, then the events A and B are not independent and the required probability is given by :

$$P(A \cap B) = P(A) \cdot P(B | A)$$

As discussed in part (i) $P(A) = \dfrac{\binom{5}{3}}{\binom{13}{3}}$ **(i)**

If 3 white balls which were drawn in the first draw are not replaced back, then there are $13 - 3 = 10$ balls left in the bag. (8 red, 2 white)

Hence, $P(B \mid A) = \dfrac{\binom{8}{3}}{\binom{10}{3}}$ (ii)

Thus, from (i) and (ii)

$$P(A \cap B) = P(A) \cdot P(B \mid A) = \frac{\binom{5}{3}}{\binom{13}{3}} \cdot \frac{\binom{8}{3}}{\binom{10}{3}} = \frac{7}{429}$$

Example 8 : A and B are two independent events such that $P(A \cup B) = 0.5$ and $P(A) = 0.2$, find $P(B)$.

Solution : Since A and B are independent events, we have $P(A \cap B) = P(A) \cdot P(B)$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)\ P(B)$

$= P(A) + P(B)\ (1 - P(A))$

$\therefore \quad 0.5 = 0.2 + P(B)\ (1 - 0.2)$

$\therefore \quad 0.3 = P(B) \times 0.8$

$\therefore \quad P(B) = \frac{3}{8}$

Example 9 : A machine manufactured by a firm consists of two parts A and B. Out of 100 A's manufactured, 9 are likely to be defective and out of 100 B's manufactured, 5 are likely to be defective. Find the probability that a machine manufactured by the firm is free of any defect.

Solution : Let event E : Part A of the machine is defective

and event F : Part B of the machine is defective.

By the given conditions,

$$P(E) = \frac{9}{100},\ P(F) = \frac{5}{100}$$

Event E' : Part A is not defective and

Event F' : Part B is not defective.

$$\therefore \quad P(E') = 1 - P(E) = 1 - \frac{9}{100} = \frac{91}{100}$$

$$P(F') = 1 - P(F) = 1 - \frac{5}{100} = \frac{95}{100}$$

Since E and F are independent events, E' and F' are also independent.

Now, machine manufactured is free of any defect is the event $E' \cap F'$.

$$\therefore \quad P(E' \cap F') = P(E') \cdot P(F')$$

$$= \frac{91}{100} \cdot \frac{95}{100} = \frac{8645}{10000} = 0.8645$$

Example 10 : A purse contains 6 silver coins and 3 gold coins. Another purse contains 4 silver coins and 5 gold coins. A purse is selected at random and a coin is drawn from it. What is the probability that it is a silver coin ?

Solution : Let the event B_1 be the first purse is selected and the event B_2 be the second purse is selected.

$\therefore$ $P(B_1) = \frac{1}{2}$ and $P(B_2) = \frac{1}{2}$

Event A : Selected coin is a silver coin.

$\therefore$ $P(A \mid B_1) = \frac{6}{9} = \frac{2}{3}$ **(Total coins 9, silver coins 6)**

Similarly, $P(A \mid B_2) = \frac{4}{9}$

$\therefore$ Required probability

$$P(A) = P(B_1)\, P(A \mid B_1) + P(B_2)\, P(A \mid B_2)$$

$$= \frac{1}{2} \times \frac{6}{9} + \frac{1}{2} \times \frac{4}{9} = \frac{10}{18} = \frac{5}{9}$$

Example 11 : In a class of 75 students, 15 students have taken AB group. 45 have taken A group and the rest of them have taken B group. The probability that an AB group student fails in a KVPY (Kishor Vigyan Protsahak Yojana) examination is 0.005; an A group student failing has a probability 0.05 and the corresponding probability for a B group student is 0.15. If a student is known to have passed the KVPY examination, what is the probability that she is a student of B group ?

Solution : Let us define the following events :

B_1 : The student has taken AB group

B_2 : The student has taken A group

B_3 : The student is of B group

A : The student passes in the KVPY examination.

By the given information :

$P(B_1) = \frac{15}{75} = 0.2$, $P(B_2) = \frac{45}{75} = 0.6$, $P(B_3) = \frac{15}{75} = 0.2$

$P(A \mid B_1) = 1 - 0.005 = 0.995$, $P(A \mid B_2) = 1 - 0.05 = 0.950$, $P(A \mid B_3) = 1 - 0.15 = 0.850$

Now, $P(A) = P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)$

$$= (0.995)(0.2) + (0.95)(0.6) + (0.85)(0.2)$$

$$= 0.1990 + 0.570 + 0.170$$

$$= 0.939 \qquad \textbf{(i)}$$

We have to find $P(B_3 \mid A)$.

By Bayes' theorem,

$$P(B_3 \mid A) = \frac{P(A \mid B_3)\, P(B_3)}{\sum_{i=1}^{3} P(A \mid B_i)\, P(B_i)}$$

$$= \frac{P(A \mid B_3)\, P(B_3)}{P(A)}$$

$$= \frac{(0.2)\,(0.850)}{0.939} \qquad \textbf{(by (i))}$$

$$= \frac{0.170}{0.939} = \frac{170}{939}$$

Exercise 7.2

1. A card is drawn from a well shuffled pack of 52 cards. Events A and B are defined as follows :

   A : getting a card of spade

   B : getting an ace

   Determine whether the events A and B are independent or not.

2. If $P(B') = 0.65$, $P(A \cup B) = 0.85$ and A and B are independent events, then find P(A).

3. 10 boys and 5 girls study in a class. Three students are selected at random, one after the other. Find probability that,

   (1) First two are boys and the third is a girl,

   (2) First and third are boys and second is a girl,

   (3) First and third are of same sex and the second is of opposite sex.

4. Police plan to enforce speed limits by using radar system at 3 different locations within the city limits. The radar system at each of these locations are operated for 40 %, 30 % and 20 % of the time. If a person who is speeding on his way to work has probabilities of 0.2, 0.1 and 0.5 respectively of passing through these locations, what is the probability that he will be fined ?

5. Suppose coloured balls are distributed in three boxes are as follows :

Colour ↓	Box 1	Box 2	Box 3
Red	2	4	3
White	3	1	4
Blue	5	3	3
Total	**10**	**8**	**10**

   A box is selected at random from which a ball is selected at random. What is the probability that the ball selected of red colour ?

6. Three machines A, B and C produce respectively 50 %, 30 % and 20 % of the total number of items of a factory. The percentage of defective output of these machines are 3 %, 4 % and 5 % respectively. If an item is selected at random, find the probability that the item is non-defective.

7. In a certain college 25 % of boys and 10 % of girls are studying mathematics. The girls constitute 60 % of the student body.

   (1) What is the probability that mathematics is being studied ?

   (2) If a student is selected at random and is found to be studying mathematics, what is the probability that the student is a girl ?

8. There are two therapies B_1 and B_2 available for curing a patient suffering from a certain disease. The patient can choose any one of the two therapies. If he selects therapy B_1 the probability of his recovery from the disease is $\frac{7}{8}$ and if he selects therapy B_2 the the probability of his recovery from the disease is $\frac{9}{10}$ (i) what is the probability that the patient is cured from the disease ? (ii) Given that the patient is cured, what is the probability that he has selected therapy B_2 ?

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7.4 Random Variable and Probability Distribution

We have studied how we can determine probability of various events using probability function defined on the power-set S of a sample space associated with all possible outcomes of a random experiment. In many real situations we are not interested in studying the details of all outcomes of a random experiment. For instance, in a sample space with possible outcomes *bb*, *bg*, *gb*, *gg* of a random experiment of having two children in a family, we are interested in knowing the number of boys (or number of girls) rather than the outcomes themselves. Similarly, in case of a randomly selected electric bulb from a lot of electric bulbs produced in a factory, we are interested in determining the life in hours. Thus, we associate a real number, in one way or another, with an outcome of each of the random experiments described above. In other words, we define a real-valued function on a sample space associated with a random experiment and we shall call this real valued function a **'random variable'**. We shall study a random variable and its probability distribution in this section.

Let us understand the idea of a random variable by considering a simple example. Suppose we select a family having two children. b represents a boy. g represents a girl. The sample space associated with the random experiment is $U = \{bb, bg, gb, gg\}$.

If the outcomes of U are equally likely, we have by the classical definition of probability,

$$P(\{bb\}) = P(\{bg\}) = P(\{gb\}) = P(\{gg\}) = \frac{1}{4}$$

Suppose $X : U \to R$ is a real valued function defined by, $X(u)$ = number of boys in u.

If $u = bb$, then $X(bb) = 2$. If $u = gg$, then $X(gg) = 0$ and for $u = bg$ or gb, $X(bg) = X(gb) = 1$.

Hence, the range of function $X : U \to R$ is the set $\{0, 1, 2\}$. We now take the subset $\{1\}$ of the range of the function X. Pre-image set of $\{1\}$ is $\{u \in U \mid X(u) = 1\} = \{bg, gb\}$.

Similarly, pre-image set of $\{2\}$ is $\{bb\}$ and pre-image set of $\{0\}$ is $\{gg\}$ and pre-image set of $\{0, 1, 2\}$ is $\{bb, bg, gb, gg\} = U$.

Thus, corresponding to any value in the set $\{0, 1, 2\}$ assumed by X there corresponds some event of sample space U.

As an example for $X(u) = 0$ for $u \in U$ there corresponds the event $\{gg\}$. Hence, the probability that $X(u) = 0$ is equal to the probability of the event $\{gg\}$. Therefore $P(X(u) = 0) = P(\{gg\}) = \frac{1}{4}$.

In the table below the values of probabilities associated with the elements of the range set of the function X are shown :

Element u of U	Probability of event $\{u\}$, $P(\{u\})$	$X(u) = x$	$P(X(u) = x)$
bb	$P(\{bb\}) = \frac{1}{4}$	2	$\frac{1}{4}$
bg	$P(\{bg\}) = \frac{1}{4}$	1	$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
gb	$P(\{gb\}) = \frac{1}{4}$		
gg	$P(\{gg\}) = \frac{1}{4}$	0	$\frac{1}{4}$

We shall call a real valued function on the sample space as a random variable, denoted by X and its value by x. The probability with which X assumes a value x shall be denoted by $p(x)$.

That is $p(x) = P(X = x) = P(X(u) = x)$

The various real values assumed by a random variable X and its corresponding probabilities, as shown in the table above, can be expressed as follows :

$X = x$	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Obviously, $\sum_{x=0}^{2} p(x) = p(0) + p(1) + p(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$

This table gives the probability distribution of random variable X and $p(x)$ is called the probability function of random variable X.

Now, we shall define a random variable X and its probability distribution.

Random Variable : Let U be the sample space associated with a random experiment. A real valued function X defined on U i.e. $X : U \to R$ is called a random variable.

There are two kinds of random variables in the study of statistics, namely discrete random variable and continuous random variable. If the range of the real function $X : U \to R$ is a finite set or an infinite sequence of real numbers, then it is called a discrete random variable. If the range of X contains interval of R, then X is called a continuous random variable. We shall consider a discrete random variable with finite range and its probability distribution only. Thus, we shall assume the range of random variable $X : U \to R$ as $\{x_1, x_2, ..., x_n\}$.

Probability Distribution of Random Variable :

Let $X : U \to R$ be a random variable. Suppose X has range $\{x_1, x_2, ..., x_n\}$ which is a subset of R. Further suppose that X assumes a value x_i with probability $p(x_i) = P(X = x_i)$.

If (i) $p(x_i) \geq 0$, $i = 1, 2, ..., n$ and (ii) $\sum_{i=1}^{n} p(x_i) = 1$, then the set $\{p(x_1), p(x_2), ..., p(x_n)\}$ is called a probability distribution of the random variable X.

We can write probability distribution of the random variable X in tabular form as follows :

$X = x$	x_1	x_2	x_3	...	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	...	$p(x_n)$

Example 12 : A random variable $X : U \to R$, where U is a sample space associated with tossing of a fair coin three times, is defined as : For $u \in U$, $X(u)$ = number of heads in u. If the outcomes of U are equally likely, then obtain probability distribution of X.

Solution : The sample space associated with tossing of a fair coin three times is

U = {HHH, HHT, HTH, THH, THT, HTT, TTH, TTT}

If u = HHH, then according to the definition of the random variable, X(HHH) = 3.

If u = HHT or HTH or THH, then $X(u) = 2$

If u = THT or HTT or TTH, then $X(u) = 1$

If u = TTT, then $X(u) = 0$

Thus, the range of random variable X is the set {0, 1, 2, 3}. Since the elementary events of U are equally likely, we have

$P(\{HHH\}) = P(\{HHT\}) = P(\{HTH\}) = P(\{THH\}) = P(\{THT\}) = P(\{HTT\}) = P(\{TTH\}) = P(\{TTT\}) = \frac{1}{8}$

The probabilities associated with various values assumed by random variable X are given in the following table :

Element u of U	Probability $P(\{u\})$	$X(u) = x$	$P(X = x)$
HHH	$\frac{1}{8}$	3	$\frac{1}{8}$
HHT	$\frac{1}{8}$		
HTH	$\frac{1}{8}$	2	$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$
THH	$\frac{1}{8}$		
TTH	$\frac{1}{8}$		
THT	$\frac{1}{8}$	1	$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$
HTT	$\frac{1}{8}$		
TTT	$\frac{1}{8}$	0	$\frac{1}{8}$

Thus, the probability distribution of the random variable X is as follows :

$X = x$	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example 13 : Four raw mangoes are mixed accidently with 16 ripe mangoes. Find the probability distribution of the number of raw mangoes in a draw of two mangoes.

Solution : Let X denote the number of raw mangoes in a draw of 2 mangoes drawn from the group of 16 ripe mangoes and 4 raw mangoes. Since there are 4 raw mangoes in the group, X can take values 0, 1 and 2.

Now, $P(X = 0)$ = Probability of getting 0 raw mango

$$= \frac{\binom{16}{2}}{\binom{20}{2}} = \frac{16 \times 15}{2} \times \frac{2}{20 \times 19} = \frac{12}{19}$$

$P(X = 1)$ = Probability of getting one raw mango

$$= \frac{\binom{4}{1}\binom{16}{1}}{\binom{20}{2}} = \frac{4 \times 16 \times 2}{20 \times 19} = \frac{32}{95}$$

and $P(X = 2)$ = Probability of getting two raw mangos

$$= \frac{\binom{4}{2}}{\binom{20}{2}}$$

$$= \frac{4 \times 3}{2} \times \frac{2}{20 \times 19}$$

$$= \frac{3}{95}$$

Thus, the probability distribution of X is given by

$X = x$	0	1	2
$p(x)$	$\frac{12}{19}$	$\frac{32}{95}$	$\frac{3}{95}$

Example 14 : Find the constant c for the probability distribution $p(x) = c\binom{5}{x}$, $x = 0, 1, 2, 3, 4, 5$

Solution : Here, $p(x) = c\binom{5}{x}$, $x = 0, 1, 2, 3, 4, 5$

Since, $p(x)$ represents probability distribution of X, we should have

$$p(0) + p(1) + p(2) + p(3) + p(4) + p(5) = 1$$

$$\therefore \quad c\left[\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right] = 1$$

$$\therefore \quad c(2^5) = 1$$

$$\therefore \quad 32c = 1$$

$$\therefore \quad c = \frac{1}{32}$$

Also, for each value of x, $p(x) > 0$.

$\therefore$ Required value of c is $\frac{1}{32}$

Example 15 : Probability distribution of a discrete random variable X is given in the following table :

$X = x$	-3	-2	-1	0	1	2	3
$p(x)$	0.08	0.14	0.19	0.27	0.17	0.09	0.06

(1) Find the probability of random variable X assuming negative values.

(2) Find the value of $P(0 \le x < 3)$.

Solution : (1) Probability that X assumes negative values is

$$p(-3) + p(-2) + p(-1) = 0.08 + 0.14 + 0.19 = 0.41$$

(2) $P(0 \le x < 3) = p(0) + p(1) + p(2)$

$= 0.27 + 0.17 + 0.09$

$= 0.53$

Exercise 7.3

1. Find the constant c for each of the following probability distribution :

(1) $p(x) = cx$, $x = 1, 2, 3, 4$

(2) $p(x) = cx^2$, $x = 1, 2, ..., 10$

(3) $p(x) = c \cdot 3^x$, $x = 0, 1, 2, 3$

(4) $p(x) = c\left(\frac{1}{4}\right)^x$, $x = 1, 2, 3$

(5) $p(x) = c\binom{4}{x}$, $x = 0, 1, 2, 3, 4$

2. Examine whether $p(x)$ defined for a random variable X as below is a probability distribution :

$$p(x) = \frac{2x}{n(n+1)}, \; x = 1, 2, 3, ..., n$$

3. Let X denote the number of hours you study during a randomly selected school day. The probability that X can take values x, has the following form, where k is some unknown constant.

$$P(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ kx, & \text{if } x = 1 \text{ or } 2 \\ k(5 - x), & \text{if } x = 3 \text{ or } 4 \\ 0, & \text{otherwise} \end{cases}$$

(1) Find the value of k.

What is the probability that you study.

(2) for at least two hours (3) for exactly two hours (4) for at most two hours ?

4. Two balanced dice are tossed once. A random variable X is defined on the sample space U associated with this random experiment as follows :

For $u \in$ U, X(u) = sum of integers in u.

Find the probability distribution of X.

5. A box contains 4 distinct balls of which 2 are white and 2 are black. Two balls are selected at random with replacement. If X denotes the number of black balls in the two balls selected from the box, then find the probability distribution of X.

6. From a lot of 10 bulbs, which includes 3 defectives bulbs, a sample of 2 bulbs is drawn at random. Find the probability distribution of the number of defective bulbs.

7. The probability distribution of a discrete random variable X is given in the following table :

X = x	0	1	2
$p(x)$	$3c^3$	$4c - 10c^2$	$5c - 1$

where $c > 0$. Find (1) c (2) $P(X < 2)$ (3) $P(1 < X \le 2)$

8. We take 8 identical slips of paper, write the number 0 on one of them, the number 1 on three of the slips, the number 2 on three of the slips and the number 3 on one of the slips. The slips are folded. Put in a box and throughly mixed. One slip is drawn at random from the box. If X is the random variable denoting the number written on the drawn slip, find the probability distribution of X.

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7.5 Mathematical Expectation

Suppose that the following is the probability distribution of a random variable X :

X = x	x_1	x_2	x_3	...	x_{n-1}	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	...	$p(x_{n-1})$	$p(x_n)$

where $p(x_i) \ge 0$ for each x_i and $\sum_{i=1}^{n} p(x_i) = 1$ (i)

Mean : X is a random variable with probability distribution given by (i). We denote mathematical expectation of X by E(X) and it is defined as :

$$E(X) = \sum_{i=1}^{n} x_i \, p(x_i) \qquad \text{(ii)}$$

Mathematical expectation of a random variable X is called the expected value of X or mean of X. E(X) is also denoted by the symbol μ. Mean of X is infact the weighted average of the possible values of X, each value being weighted by its probability with which it occurs.

Suppose Y = g(X) is a real function of a discrete random variable X. Then Y = g(X) will also be a discrete random variable and its mean is defined as

$$E(Y) = E[g(X)] = \sum_{i=1}^{n} g(x_i) \, p(x_i) \qquad \text{(iii)}$$

e.g. if $g(X) = X^2$, then

$$E[g(X)] = E(X^2) = \sum_{i=1}^{n} x_i^2 \, p(x_i) \qquad \text{(iv)}$$

Variance of Random Variable :

The mean or expected value of a random variable X is of special importance in statistics because it describes where the probability distribution is centered. However, the only mean does not give adequate description of the shape of the distribution. We need to characterise the variability in the distribution. In figure 7.2 we have the histograms of two discrete probability distributions with the same mean $\mu = 2$ that differ considerably in the variability of their observations about the mean.

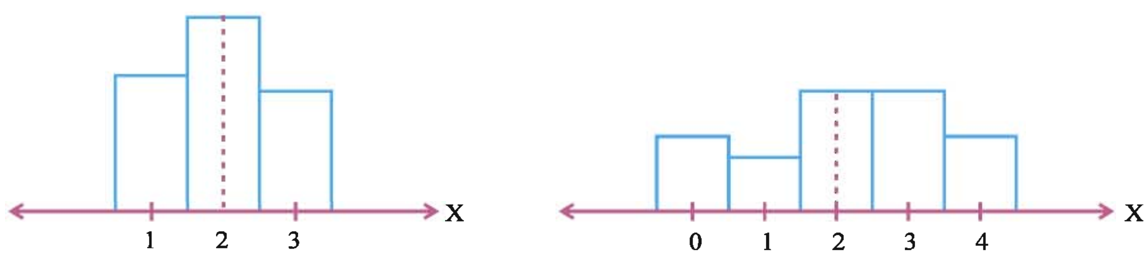

Fig. 7.2

The most important measure of variability of random variable X is referred to as the variance of the random variable X. We shall denote it by the symbol σ_X^2 or V(X). If the probability distribution of a random variable X is given by (i), then variance of X is defined by :

$$V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$$

Using formula (ii) and (iv) the formula for σ_X^2 is written as

$$\sigma_X^2 = \sum_{i=1}^{n} x_i^2 \, p(x_i) - \left[\sum_{i=1}^{n} x_i \, p(x_i)\right]^2 \qquad \text{(v)}$$

Standard Deviation of Random Variables :

The positive square root of variance σ_X^2 of a random variable X is called the standard deviation of X. It is denoted by the symbol σ_X or $\sqrt{V(X)}$.

Some Results About Mathematical Expectation :

Suppose that the mathematical expectation and variance of a random variable X are E(X) and σ_X^2 respectively. For real constants a, b and c, let $Y = aX + b$ and $Z = aX^2 + bX + c$ be the new random variables. We shall assume the following results on expectation without proof.

$$E(Y) = E(aX + b) = aE(X) + b \quad \text{(vi)}$$

$$\sigma_Y^2 = V(Y) = V(aX + b) = a^2 V(X) = a^2 \sigma_X^2 \quad \text{(vii)}$$

$$\sigma_Y = \sqrt{V(Y)} = |a| \sigma_X \quad \text{(viii)}$$

$$E(Z) = E(aX^2 + bX + c) = aE(X^2) + bE(X) + c \quad \text{(ix)}$$

Example 16 : Probability distribution of a random variable X is as follows :

X = x	–2	–1	0	1	2	3
$p(x)$	0.05	0.14	0.23	0.31	0.16	0.11

Find E(X) and σ_X.

Solution : $E(X) = \sum x_i\, p(x_i)$

$= (-2)(0.05) + (-1)(0.14) + (0)(0.23) + (1)(0.31) + (2)(0.16) + (3)(0.11)$

$= -0.10 - 0.14 + 0 + 0.31 + 0.32 + 0.33$

$= 0.72$

$\therefore \quad E(X) = 0.72$

$\sigma_X^2 = \sum_i x_i^2\, p(x_i) - [E(X)]^2$

$= \{4(0.05) + 1(0.14) + 0(0.23) + 1(0.31) + 4(0.16) + 9(0.11)\} - (0.72)^2$

$= 2.28 - 0.5184 = 1.7616$

$\therefore \quad \sigma_X^2 = 1.7616$ and

$\sigma_X = \sqrt{1.7616} = 1.3272$

Example 17 : The mean and the standard deviation of a random variable X are given by E(X) = 5 and $\sigma_X = 3$ respectively. Find $E(X^2)$, $E((3X + 2)^2)$. Also find the standard deviation of $2 - 3X$.

Solution : Here, E(X) = 5 and $\sigma_X = 3$

We know that, $\sigma_X^2 = E(X^2) - [E(X)]^2$

$\therefore \quad E(X^2) = \sigma_X^2 + [E(X)]^2$

$= 9 + 25$

$E(X^2) = 34$

$E((3X + 2)^2) = E(9X^2 + 12X + 4)$

$= 9E(X^2) + 12E(X) + 4$

$= 9 \cdot 34 + 12 \cdot 5 + 4$

$= 306 + 60 + 4$

$E((3X + 2)^2) = 370$

Now, $V(2 - 3X) = 3^2V(X) = 9V(X) = 9\sigma_X^2 = 9 \cdot 9 = 81$

$\therefore$ The standard deviation of $2 - 3X$ is $\sqrt{81} = 9$.

If the expected gain of two players playing a game is zero, then the game is said to be fair. If the expected gain of any player is positive, the game is said to be in his favour. If the expected gain of a player is negative the game is said to be against him.

Example 18 : A player playing a game of tossing a balanced die receives ₹ 10 from his opponent if he throws an integer 3 or 4. If he throws 1 or 2 or 5 or 6, then how much should be pay to his opponent, so that the game becomes fair ?

Solution : Sample space associated in the game of tossing a die is U = {1, 2, 3, 4, 5, 6}. We define a random variable X on U as follows :

$$X(u) = \begin{cases} 10 & u = 3, 4 \\ a & u = 1, 2, 5, 6 \end{cases}$$

where a is the amount in rupees which the player has to pay to his opponent.

The probability distribution of X is as follows :

$X = x$	10	a
$p(x)$	$\frac{2}{6}$	$\frac{4}{6}$

Now, $E(X) = 10 \cdot \frac{2}{6} + a \cdot \frac{4}{6}$

$= \frac{4a + 20}{6}$

Since the game is to be fair, we must have $E(X) = 0$.

$\therefore \quad \frac{4a + 20}{6} = 0$

$\therefore \quad 4a + 20 = 0$

$\therefore \quad a = -5$

Hence, the player has to pay ₹ 5 to his opponent if $u = 1, 2, 5$ or 6.

Exercise 7.4

1. Determine the discrete probability distribution, mathematical expectation, variance, standard deviation of a discrete random variable X which denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once.

2. A player tosses 3 fair coins. He wins ₹ 500 if 3 heads occur, ₹ 300 if 2 heads occur, ₹ 100 if one head occurs. On the other hand, he loses ₹ 1500 if 3 tails occur. Find the expected value of the game for the player. Is it favourable to him ?

3. The probability distribution of a random variable X is as follows :

$X = x$	1	2	3	4	k
$p(x)$	0.1	k	0.2	$3k$	0.3

(1) Find the value of k.

(2) Find the mean and variance.

4. The probability distribution of a random variable X is as follows :

$X = x$	-1	0	1	2	3
$p(x)$	0.2	0.1	k	$2k$	0.1

(1) Find the value of k.

(2) Calculate the mean, variance and standard deviation.

5. Find the variance of the numbers obtained at the throw of an unbaised die.

6. Probability distribution of a random variable X is as follows :

$X = x$	-2	-1	0	1	2
$p(x)$	0.2	0.1	0.3	0.3	0.1

Find (1) E(X) (2) V(X) (3) E(3X + 2) (4) V(3X + 2)

7. A bakery owner finds from his past experience that sale of number of chocolate cakes produced in his bakery on any day is a random variable X having the following probability distribution :

No. of cakes sold $X = n$	0	1	2	3	4	5
$p(n)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

He gets a profit of ₹ 5 per each cake sold and incurs a loss of ₹ 2 per cake not being sold. If the bakery owner produces 3 cakes on a given day what is the value of his expected profit ?

8. The mean and standard deviation of a random variable X are 10 and 5 respectively. Find $E(X^2)$, $E[X(X + 1)]$, $E\left(\frac{X - 10}{5}\right)$ and $E\left(\frac{X - 10}{5}\right)^2$.

*

7.6 Binomial Distribution

We have studied a random variable and its probability distributions in the earlier sections of this chapter. In this section we shall study a special distribution, a **binomial distribution.**

Binomial distribution is also known as the **'Bernolli distribution'** after the Swiss mathematician **James Bernoulli (1654-1705)** who discovered it in 1700.

Let us consider an experiment of tossing a coin. If we toss a coin, we get two outcomes namely, 'Head' or 'Tail'. For the sake of definiteness we shall call 'Head' **a success** and 'Tail' **a failure**. Hence sample space associated with the experiment is U = {S, F} where S denotes success and F denotes failure. Suppose that probability of getting a success is p and that of getting failure is q. That is $P(\{S\}) = p$ and $P(\{F\}) = q$. Since there are two outcomes of the experiment we must have $p + q = 1$ and hence $q = 1 - p$.

Suppose a coin is tossed n times under identical conditions. Alternatively we can say that an experiment of tossing a coin is repeated n times under identical conditions. Since the experiment is performed under identical conditions, the probability of getting success 'S' at each of the n trials remains the same i.e., p. Trials of a random experiment possessing this property are called **Bernoulli trials**. We now define Bernoulli trials as follows :

Bernoulli trials : Suppose a random experiment has two possible outcomes namely success (S) and failure (F). If the probability p $(0 < p < 1)$ of getting success at each of the n trials of this experiment is constant, then the trials are called Bernoulli trials.

Bernoulli trials have following properties :

(1) **There is a constant probability of success (S) or failure (F) at each Bernoulli trial.**

(2) **Bernoulli trials are mutually independent**

(3) **If the constant probability of getting a success (S) at any Bernoulli trial is p $(0 < p < 1)$, then probability of getting a failure (F) is $q = 1 - p$.**

Suppose X denotes number of successes in a sequence of n Bernoulli trials of a random experiment having a constant probability p of success. Suppose that the probability distribution of random variable X is given by

$$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, ... n \qquad \text{(i)}$$

where $0 < p < 1$ and $q = 1 - p$

Probability distribution of random variable X given by (i) is called a Binomial distribution and random variable X is called a binomial random variable. The positive integer n and probability p of success 'S' are called the parameters of the binomial distribution.

The formula of $p(x)$ given by (i) for $x = 0, 1, 2, ..., n$ can be obtained from the binomial expansion of $(p + q)^n$. The general term of the binomial expansion of $(p + q)^n$ is $\binom{n}{x} p^x q^{n-x}$ which is equal to the formula (i). Hence, the probability distribution of random variable is called the binomial distribution. Also, sum of all probabilities is

$$\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p + q)^n = 1^n = 1$$

The binomial distribution occurs in games of chance (e.g. rolling a dice), quality inspection (e.g. count of number of defectives), opinion polls (e.g. number of empolyees favouring certain schedule changes), medicine (e.g. number of patients recovered by a new medication) and so on.

Result : The mean and variance of binomial distribution with parameters n and p are np and npq respectively.

Example 19 : It has been claimed that in 60 % of all solar-light installations, the utility bill is reduced by at least one third. Accordingly, what are the probabilities that the utility bill will be reduced by at least one third in

(1) four of five installations;

(2) at least four of five installations.

Solution : Let X denote the number of solar-light installations in which the utility bill is reduced by at least one third out of 5 solar-light selected at random from a lot.

Here, X is a binomial random variable having binomial distribution with parameters $n = 5$ and $p = 0.60$. Hence, the probability distribution of X is given by

$$p(x) = \binom{5}{x} \left(\frac{6}{10}\right)^x \left(\frac{4}{10}\right)^{5-x}, \; x = 0, 1, 2, 3, 4, 5$$

(1) The probability of the utility bill is reduced by at least one third in four installations by putting $x = 4$ in $p(x)$.

$\therefore \quad p(4) = \binom{5}{4}\left(\frac{6}{10}\right)^4\left(\frac{4}{10}\right)^{5-4}$

$= 5(0.6)^4\ (0.4)$

$= 0.2592$ **(i)**

(2) The probability that utility bill is reduced by at least one third in at four installations is $p(4) + p(5)$. Now,

$p(5) = \binom{5}{5}\left(\frac{6}{10}\right)^5\left(\frac{4}{10}\right)^{5-5}$

$= (0.6)^5$

$= 0.07776$

Hence, required probability $= p(4) + p(5)$

$= 0.2592 + 0.07776$ **(by (i))**

$= 0.337$

Example 20 : The mean and variance of a binomial distribution are 3 and 2 respectively. Find the probability that the variate takes values less than or equal to 2.

Solution : If n and p are the parameters of the binomial distribution, then we know that

Mean $= np = 3$ **(i)**

and Variance $= npq = 2$ **(ii)**

Dividing (ii) by (i) we get, $\frac{npq}{np} = \frac{2}{3}$

$\therefore \quad q = \frac{2}{3}$. So, $p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

Substituting in (i) we get, $n \cdot \frac{1}{3} = 3$. So, $n = 9$

$\therefore$ The probability distribution of binomial random variable X is given by

$p(x) = \binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}, x = 0, 1, 2, ..., 9$

The probability that the variable takes the value less than or equal to 2 is given by $P(X \le 2)$.

$P(x \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$= p(0) + p(1) + p(2)$

$= \binom{9}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 + \binom{9}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8 + \binom{9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^7$

$= \left(\frac{2}{3}\right)^7\left[\left(\frac{2}{3}\right)^2 + 9 \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{9 \cdot 8}{2}\left(\frac{1}{3}\right)^2\right]$

$= \left(\frac{2}{3}\right)^7\left[\frac{4}{9} + 2 + 4\right] = \left(\frac{2}{3}\right)^7 \frac{58}{9} = \left(\frac{2^7}{3^9}\right)58$

Exercise 7.5

1. An educationist claims that 80 percent of the students passing a higher secondary examination take admission to colleges for university education. What is the probability that out of 10 students (1) 5 students (2) 8 or more students take admission to a college ?
2. It has been found from an experiment that 40 percent of rats get stimulated on administering a particular drug. If 5 rats are given this drug, what is the probability that (1) exactly three and (2) all rats get stimulated ?
3. In a city of some western country, 70 percent of the married persons take divorce. What is the probability that at least three among four persons will take divorce ?
4. Harit participates in a shooting competition. The probability of his shooting a target is 0.2. What is the probability of shooting the target exactly three times out of five trials ?
5. The mean and standard deviation of a binomial random variable X are 8 and 2 respectively. Find the parameters of the probabilty distribution of X and obtain the value of $P(X = 0)$ and $P(1 \le X \le 3)$.
6. In a book of 500 pages, there are 50 printing errors. Find the probability of at most two printing errors in 4 pages selected at random from the book.
7. If 4 of 12 scooterists do not carry driving licence, what is the probability that a traffic inspector who randomly selects 4 scooterists will catch (1) 1 for not carrying driving licence. (2) at least 2 for not carrying driving licence.
8. In a shooting competition, the probability of a man hitting a target is $\frac{2}{5}$. If he fires 5 times, what is the probability of hitting the target (1) at least twice (2) at most twice.
9. A quality control engineer inspects a random sample of 3 calculators from a lot of 20 calculators. If such a lot contains 4 slightly defective calculators, what is the probability that the inspector's sample will contain (1) no slightly defective calculators, (2) one slightly defective calculators, (3) at least two slightly defective calculators.
10. If the probability of selecting a defective bolt is 0.1, find (1) the mean (2) the variance for the distribution of defective bolts in a total of 400.

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Miscellaneous Examples :

Example 21 : Suppose E and F be two independent events such that $P(E) < P(F)$. If $P(E \cap F) = \frac{1}{12}$ and $P(E' \cap F') = \frac{1}{2}$, then find P(E) and P(F).

Solution : We are given $P(E \cap F) = \frac{1}{12}$ and $P(E' \cap F') = \frac{1}{2}$.

As E and F are independent events, E' and F' are also independent events.

$P(E \cap F) = \frac{1}{12} \Rightarrow P(E)\,P(F) = \frac{1}{12}$ and

$P(E' \cap F') = \frac{1}{2} \Rightarrow P(E')\,P(F') = \frac{1}{2}$

$\therefore \quad [1 - P(E)]\,[1 - P(F)] = \frac{1}{2}$

$\therefore \quad 1 - P(E) - P(F) + P(E)\,P(F) = \frac{1}{2}$

$\therefore \quad 1 - P(E) - P(F) + \frac{1}{12} = \frac{1}{2}$

$\therefore \quad P(E) + P(F) = 1 + \frac{1}{12} - \frac{1}{2}$

$\therefore \quad P(E) + P(F) = \frac{7}{12}$

We know that the quadratic equation whose roots are a and b is $x^2 - (a + b)x + ab = 0$

$\therefore$ The equation whose roots are P(E) and P(F) is

$x^2 - [P(E) + P(F)]x + P(E)\ P(F) = 0$

$\therefore \quad x^2 - \frac{7}{12}x + \frac{1}{12} = 0$

$\therefore \quad 12x^2 - 7x + 1 = 0$

$\therefore \quad (3x - 1)(4x - 1) = 0$

$\therefore \quad x = \frac{1}{3}, \frac{1}{4}$

Since $P(E) < P(F)$, we have $P(E) = \frac{1}{4}$ and $P(F) = \frac{1}{3}$.

Example 22 : Find the number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.95.

Solution : Let n be the required number of tosses, and X be the number of heads obtained in n tosses. Then X is a binomial random variable having binomial distribution with parameters n and $p = \frac{1}{2}$. Hence, the probability distribution of X is given by

$$p(x) = \binom{n}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{n-x}, \ x = 0, 1, 2, ..., n$$

Now, P(at least one head) $= P(X \geq 1)$

$= 1 - P(X = 0)$

$= 1 - p(0)$

$= 1 - \binom{n}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{n-0}$

$= 1 - \left(\frac{1}{2}\right)^n$

Given P(at least one head) ≥ 0.95

$\therefore \quad 1 - \left(\frac{1}{2}\right)^n \geq 0.95$

$\therefore \quad \left(\frac{1}{2}\right)^n \leq 0.05$

$\therefore \quad \frac{1}{2^n} \leq \frac{1}{20}$

$\therefore \quad 2^n \geq 20$

$\therefore \quad n \geq 5$

$\therefore$ The least value of n is 5.

Hence, under the given conditions a fair coin must be tossed at least 5 times.

Example 23 : For a random experiment the sample space is U = {(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)}.

Events A, B, C are defined as follows :

A = {(0, 0, 0), (1, 0, 0)}, B = {(0, 0, 0), (0, 1, 0)}, C = {(0, 0, 0), (0, 0, 1)}

Prove A, B, C are pairwise independent but not mutually independent.

Solution : Here, $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$

$A \cap B = B \cap C = A \cap C = \{(0, 0, 0)\} = A \cap B \cap C$

$\therefore \quad P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{4} = P(A \cap B \cap C)$

Now, $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)\ P(B)$

$P(B \cap C) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(B)\ P(C)$

$P(A \cap C) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)\ P(C)$

$\therefore$ A, B, C are pairwise independent events.

But $P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A)\ P(B)\ P(C)$

$\therefore$ A, B, C are not mutually independent.

Note : If we select any vertex of tetrahedron OABC randomly, then sample space

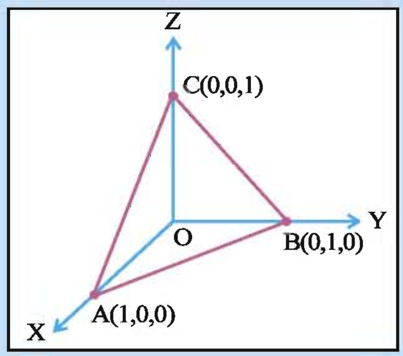
U = {(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)}.

Event A : Vertex is on X-axis.

Event B : Vertex is on Y-axis.

Event C : Vertex is on Z-axis.

Events A, B, C are as in Example 23.



Exercise 7

1. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the card drawn is more than 3, what is the probability that it is an even number ?

2. A couple has 2 children. Find the probability that both are boys, if it is known that (1) one of the children is a boy; (2) the older child is a boy.

3. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both the balls drawn are black ?

4. An urn contains 4 red and 7 blue balls. Two balls are drawn at random with replacement. Find the probability of getting (1) both red balls (2) both blue balls (3) one red and one blue ball.

5. A can hit a target 4 times in 5 shots, B can hit it 3 times in 4 shots and C can hit it 2 times in 3 shots. Calculate the probability that,

(1) A, B, C all can hit the target.

(2) B, C can hit and A cannot hit.

(3) Any two of A, B and C will hit the target

(4) None of them will hit the target.

6. A general insurance company insuring vehicles for a period of one year classifies its policy holders into three mutually exclusive group.

Group T_1 : Policy holders with very high risk factor

Group T_2 : Policy holders with high risk factor

Group T_3 : Policy holders with less rick factor

From the past experience of the company, 30 % of its policy holders belong to group T_1, 50 % belong to group T_2 and the rest belong to group T_3. If the probabilities that policy holders belonging to groups T_1, T_2 and T_3 meet with an accident are 0.30, 0.15 and 0.05 respectively, find the proportion of policy holders having a policy for one year will meet with an accident. If a randomly selected policy holder does not meet with an accident, what is the probability that he belongs to group T_2?

7. Rajesh agrees to play a game of tossing a balanced die. If an integer 1 or 2 is obtained on the die, he loses ₹ 2. If an integer 3 or 4 or 5 is obtained, he gets ₹ 5 and if integer 6 is obtained, he gets ₹ 10. If the amount of ₹ X received by Rajesh is treated as a random variable, then obtain probability distribution of X.

8. A random variable X assumes integral values from integers 1 to 100 with equal probability. Find E(X), E(X^2) and σ_X^2.

9. Nine balanced coins are tossed together once. Find probability of getting (1) four heads and (2) at least six heads.

10. The probability function of a binomial distribution is

$$p(x) = \binom{6}{x} p^x q^{6-x}, \ x = 0, 1, 2, ..., 6.$$

If $3p(2) = 2p(3)$, then find the value of p.

11. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive ? (Assume that the individual entries of the determinant are chosen independently.)

12. A restaurant serves two special dishes – A and B to its customers consisting of 60 % men and 40 % women. 80 % of men order dish A and the rest order B. 70 % of woman order B and the rest order A. In what ratio of dishes A to B should the restaurant prepare the two dishes ?

13. In a railway reservation office, two clerks are engaged in checking reservation forms. On an average, the first clerk checks 55 % of the forms, while the second checks the remaining. The first clerk has an error rate of 0.03 and second has an error rate of 0.02. A reservation form is selected at random from the total number of forms checked during a day, and is found to have an error. Find the probability that it was checked by the second clerk.

14. A fair coin tossed two times. Events A, B, C defined as follows :

Event A : First toss shows head

Event B : Second toss shows head

Event C : Same result on both toss

Show that events A, B, C are pairwise independent but not mutually independent.

15. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A : (1 mark)

(1) Two cards are drawn in succession from a standard well shuffled pack of 52 cards. What is the probability that both the cards are aces if the cards are drawn without replacement ? ☐

(a) 0.0045 (b) 0.0385 (c) 0.045 (d) 0.0059

(2) A circular wheel with numbers 1 to 20 on its surface is rolled twice. What is the probability of getting two 13's ? ☐

(a) $\frac{1}{20}$ (b) $\frac{1}{40}$ (c) $\frac{1}{400}$ (d) $\frac{1}{200}$

(3) Let A and B be two events such that P(A) = 0.4, P(A ∪ B) = 0.7 and P(B) = p. For what choice of p are A and B independent ? ☐

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{5}{6}$

(4) Two unbiased coins are tossed. If one coin shows head, the probability that the other also shows head is... ☐

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) 1

(5) A problem in mathematics is given to three students A, B, C and their respective probability of solving the problem is $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Probability that the problem is solved is... ☐

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

(6) A die is tossed 5 times. Getting an odd number is considered a success. Then the variance of distribution of success is... ☐

(a) $\frac{8}{3}$ (b) $\frac{3}{8}$ (c) $\frac{4}{5}$ (d) $\frac{5}{4}$

(7) The probability that A speaks truth is $\frac{4}{5}$, while this probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on an event is... ☐

(a) $\frac{7}{20}$ (b) $\frac{1}{5}$ (c) $\frac{3}{20}$ (d) $\frac{4}{5}$

(8) If A and B are two events such that P(A) > 0 and P(B) ≠ 1, then $P(A \mid B')$ is... ☐

(a) $1 - P(A \mid B')$ (b) $1 - P(A \mid B)$ (c) $\frac{P(A')}{P(B)}$ (d) $1 - P(A' \mid B')$

(9) The probability that a student is not a swimmer is $\frac{4}{5}$. The probability that out of 5 students exactly 4 are swimmers is...

(a) $\left(\frac{1}{5}\right)^3$ (b) $4\left(\frac{1}{5}\right)^4$ (c) ${}_5C_4\left(\frac{4}{5}\right)^4$ (d) $\left(\frac{4}{5}\right)^4$

(10) Let X be a random variable with probability distribution

X = x	0	1	2	3
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Then E(2X + 3) is...

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) 6

Section B : (2 marks)

(11) A study has been done to determine whether or not a certain drug leads to an improvement in symptoms for patients with a particular medical condition. The results are shown in the following table :

	Improvement	No improvement	Total
Drug	270	530	800
No drug	120	280	400
Total	390	810	1200

Based on this table, what is the probability that a patient shows improvement if it is known that the patient was given the drug ?

(a) 0.4375 (b) 0.225 (c) 0.3375 (d) 0.3205

(12) Suppose it is known that the patient shows improvement. Based on the table of example 11, what is the probability that the patient was given the drug ?

(a) 0.225 (b) 0.667 (c) 0.792 (d) 0.692

(13) A box contains four red, two white and three green marbles, all of which are the same size. Two marbles are selected one after the other from the box, without replacement. What is the probability that the marbles are of the same colour ?

(a) 0.67 (b) 0.5 (c) 0.14 (d) 0.28

(14) A company has three plants at which it produces a certain item. 30 % are produced at plant A, 50 % at plant B and remaining at plant C. Suppose that 1 %, 4 % and 3 % of the items produced at plants A, B and C respectively are defective. If an item is selected at random from all of those produced, what is the probability that the item was produced at plant B and is defective ?

(a) 0.5 (b) 0.2 (c) 0.02 (d) 0.04

(15) The mean and variance of a random variable X having a binomial distribution are 4 and 2 respectively, then P(X = 1) is...

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{32}$

(16) It is given that the events A and B are such that $P(A) = \frac{1}{4}$, $P(A \mid B) = \frac{1}{2}$, $P(B \mid A) = \frac{2}{3}$. Then P(B) is...

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

(17) If two events A and B are such that $P(A') = 0.3$, $P(B) = 0.5$ and $P(A \cap B) = 0.3$, then $P(B \mid A \cup B')$ is...

(a) 0.375 (b) 0.32 (c) 0.31 (d) 0.28

(18) If parameters of a binomial distribution are $n = 5$ and $p = 0.30$, then the mean is ...... and variance is ....

(a) 1.5, 1.5 (b) 1.5, 1.05 (c) 1.5, 1.40 (d) 1.5, 1.15

Section C : (3 marks)

(19) A company has three plants at which it produces a certain item. 30 % are produced at plant A, 50 % at plant B and 20 % at plant C. Suppose that 1 %, 4 % and 3 % of the items produced at plants A, B and C respectively are defective. If an item is selected at random from all those produced, what is the probability that the item is defective ?

(a) 0.029 (b) 0.29 (c) 0.025 (d) 0.08

(20) The probability that an event A occurs in a single trial of an experiment is 0.4. Three independent trials of the experiment are performed. The probability that A occurs at least once is...

(a) 0.936 (b) 0.784 (c) 0.904 (d) 0.874

(21) The variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

$X = x$	0	1	2	3
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

is...

(a) 6 (b) 36 (c) 4 (d) 8

Section D : (4 marks)

(22) A random variable X has the probability distribution :

X	1	2	3	4	5	6	7	8
$p(x)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events E = {X is a prime number} and F = {X < 4}, the probability $P(E \cup F)$ is...

(a) 0.35 (b) 0.77 (c) 0.87 (d) 0.50

(23) If a random variable X can take all non-negative integral values and the probability that X takes the value r is proportional to α^r $(0 < \alpha < 1)$, then $P(X = 0)$ is...

(a) $1 - \alpha$ (b) α (c) $\frac{\alpha}{2}$ (d) α^2

(24) The mean and standard deviation of a random veriable X are 10 and 5 respectively. Match the following : ☐

	A		B
(i)	$E(X^2)$	(p)	0
(ii)	$E(X(X+1))$	(q)	135
(iii)	$E\left(\left(\frac{X-10}{5}\right)\right)$	(r)	125
(iv)	$E\left(\left(\frac{X-10}{5}\right)^2\right)$	(s)	1

(a) (i) : (q), (ii) : (r), (iii) : (p), (iv) : (s)
(b) (i) : (r), (ii) : (q), (iii) : (s), (iv) : (p)
(c) (i) : (r), (ii) : (q), (iii) : (p), (iv) : (s)
(d) (i) : (p), (ii) : (q), (iii) : (r), (iv) : (s)

*

Summary

We studied the following points in this chapter :

1. The conditional probability of an event A, given the occurrence of the event B is given by

   $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$.

2. $0 \le P(A \mid B) \le 1$, $P(A' \mid B) = 1 - (A \mid B)$

   $P((A \cup B) \mid C) = P(A \mid C) + P(B \mid C) - P((A \cap B) \mid C)$

3. $P(A \cap B) = P(A) \cdot P(B \mid A)$, $P(A) \neq 0$

   $P(A \cap B) = P(B) \cdot P(A \mid B)$, $P(B) \neq 0$

4. If B_1 and B_2 are mutually exclusive and exhaustive events and $P(B_1) \neq 0$, $P(B_2) \neq 0$, then for any event A of S,

   $P(A) = P(B_1) P(A \mid B_1) + P(B_2) P(A \mid B_2)$

5. If B_1 and B_2 are mutually exclusive and exhaustive events and A is any event such that

   $P(A) \neq 0$, then $P(B_i \mid A) = \frac{P(A \mid B_i)\, P(B_i)}{P(A \mid B_1)\, P(B_1) + P(A \mid B_2)\, P(B_2) + P(A \mid B_3)\, P(B_3)}$, $i = 1, 2, 3$

6. If A and B are independent events then $P(A \cap B) = P(A)\, P(B)$

7. If A and B are independent events then A and B', A' and B and A' and B' are also independent.

8. A random variable is a real valued function whose domain is the sample space of a random experiment.

9. The probability distribution of a random variable X in tabular form is

$X = x$	x_1	x_2	x_3	...	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	...	$p(x_n)$

10. Mean : $E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$

Variance : $V(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$

$$\sigma_X^2 = \sum_{i=1}^{n} x_i^2\, p(x_i) - \left[\sum_{i=1}^{n} x_i\, p(x_i)\right]^2$$

Standard deviation : $\sigma_X = \sqrt{V(X)}$

11. $E(aX + b) = aE(X) + b$

12. $V(aX + b) = a^2\, V(X)$

13. Bernoulli Trials :

(1) There is a constant probability of success (S) or failure (F) at each Bernoulli trial.

(2) Bernoulli trials are mutually independent

(3) If the constant probability of getting a success (S) at any Bernoulli trial is p $(0 < p < 1)$, then probability of getting a failure (F) is $q = 1 - p$.

14. Binomial Distribution : Suppose X denotes number of successes in a sequence of n Bernoulli trials of a random experiment having a constant probability p of success. The probability distribution of random variable X is given by

$$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, 2, \ldots n$$

where $0 < p < 1$ and $q = 1 - p$ is a binomial distribution with parameters n and p.

15. The mean μ and variance σ_X^2 of binomial distribution with parameters n and p are np and npq respectively.

Ramanujan's notebooks

While still in Madras, Ramanujan recorded the bulk of his results in four notebooks of loose leaf paper. These results were mostly written up without any derivations. This is probably the origin of the misperception that Ramanujan was unable to prove his results and simply thought up the final result directly. Mathematician Bruce C. Berndt, in his review of these notebooks and Ramanujan's work, says that Ramanujan most certainly was able to make the proofs of most of his results, but chose not to.

This style of working may have been for several reasons. Since paper was very expensive, Ramanujan would do most of his work and perhaps his proofs on slate, and then transfer just the results to paper. Using a slate was common for mathematics students in the Madras Presidency at the time. He was also quite likely to have been influenced by the style of G. S. Carr's book studied in his teenage, which stated results without proofs. Finally, it is possible that Ramanujan considered his workings to be for his personal interest alone; and therefore only recorded the results.

The first notebook has 351 pages with 16 somewhat organized chapters and some unorganized material. The second notebook has 256 pages in 21 chapters and 100 unorganised pages, with the third notebook containing 33 unorganised pages. The results in his notebooks inspired numerous papers by later mathematicians trying to prove what he had found. Hardy himself created papers exploring material from Ramanujan's work as did G. N. Watson, B. M. Wilson, and Bruce Bert. A fourth notebook with 87 unorganised pages, the so-called "lost notebook", was rediscovered in 1976 by George Andrews.