LINEAR PROGRAMMING 8

Nature is an infinite sphere of which the centre is everywhere and the circumference is nowhere.

– Blaise Pascal

In order to translate a sentence from English to French, two things are necessary. First we must understand thoroughly the English sentence. Second we must be familiar with the forms of expression peculiar to French language. The situation is very similar when we attempt to express in mathematical symbols a condition proposed in words. First we must understand thoroughly the condition. Second we must be familiar with the forms of mathemaical expression.

– George Polya

8.1 Introduction

Before discussing the basic concepts and applications of linear programming, let us understand the meaning of the words, **'linear'** and **'programming'**. The word linear refers to linear relationship among variables in a model. Thus, a given change in one variable will always result into a proportional change in another variable. For example, doubling the investment on a certain investment will exactly double the return. The word programming refers to the modelling (plan of action) and solving a problem mathematically. Linear Programming was first developed by Leonid Kantorvich, a Russian mathematician, in 1939. During world war II, George B Dentzing while working with the US Air Force, developed linear programming model, primarily for solving military logistics problems.

In earlier classes, we have discussed system of linear equations and their applications in some practical problems. In class XI we have studied linear inequalities and system of linear inequalities in two variables and their solutions by graphical method. In this chapter, we shall apply the system of linear inequalities to some real life problems. The type of problems which seek to maximize (or minimize) profit (or loss) form a general class of problems called **Optimisation problems**. Any optimisation problem may involve finding maximum profit, minimum cost, or minimum use of resources etc.

A special but a very important class of optimisation problems is **Linear Programming Problems**.

Linear programming problems are of much interest because they are being used extensively in all functional areas of management, airlines, agriculture, military operations, oil refining, education, energy planning, pollution control, transportation planning and scheduling, research and development, health care system etc.

In this chapter, we shall study some linear programming problems and their solutions by graphical method only. There are many other methods also to solve such problems.

8.2 A Linear Programming Problem and its Mathematical Formulation

We begin our discussion with the help of an example which will lead us to a mathematical formulation of the problem in two variables.

A dealer deals in only two items : AC (Air conditioners) and Coolers. He has capital finance ₹ 5,00,000 to invest and has storage space of at most 60 pieces. An AC costs ₹ 25,000 and a cooler costs ₹ 5000. He estimates that from the sale of one AC, he can make profit of ₹ 2500 and from the sale of one cooler he can make profit of ₹ 750. The dealer wants to know how many AC and coolers he should buy from the available capital so as to maximise his total profit, assuming that he can sell all the items which he buys.

In this example, we observe that,

(1) The dealer can invest his money in buying AC or coolers or a combination thereof. Further he would earn different profits by following different investment strategies.

(2) There are certain **constraints** namely, his investment is limited to a maximum of ₹ 5,00,000 and storage capacity for a maximum of 60 pieces.

Suppose he decides to buy AC only and no collers, so he can buy 5,00,000 ÷ 25,000 = 20 AC. His profit in this case will be ₹ (2500 × 20) = ₹ 50,000.

Suppose he decides to buy coolers only and no AC. With his capital of ₹ 5,00,000 he can buy 100 coolers. But he can store only 60 pieces. Therefore, he has to buy only 60 coolers which will give him a total profit of ₹ (60 × 750) = ₹ 45,000.

There are many other possibilities, for instance, he may buy 10 AC and 50 coolers, as he can store only 60 pieces. Total profit in this case would be ₹ (10 × 2500 + 50 × 750) = ₹ 62,500 and so on. This, dealer can earn different profits by following different investment strategies. So, the problem is : How should the dealer invest his money in order to get maximum profit ? To answer this question, let us try to formulate the problem mathematically.

Mathematical formulation of the problem :

Let x be the number of AC and y be the number of coolers that the dealer buys.

Obviously, $x \geq 0, y \geq 0$ **(non-negative constraints) (i)**

Here, the cost of one AC is ₹ 25,000 and cost of one cooler is ₹ 5000. The dealer can invest at the most ₹ 5,00,000. Mathematically,

$$25{,}000\,x + 5000\,y \leq 5{,}00{,}000$$

$\therefore \quad 5x + y \leq 100$ **(investment constraint) (ii)**

The dealer can store maximum 60 items.

$\therefore \quad x + y \leq 60$ **(storage constraint) (iii)**

The dealer wishes to invest in such a way that he can earn maximum profit, say z.

It is given that the profit earn on selling of an AC is ₹ 2500 and that on a cooler is ₹ 750. So the profit function z is given by

$z = 2500x + 750\,y$ **(called objective function) (iv)**

Mathematically, the given problem now reduces to :

Maximise $z = 2500\,x + 750\,y$

Subject to the constraints :

$$5x + y \leq 100$$
$$x + y \leq 60$$
$$x \geq 0, y \geq 0$$

So, we have to maximise a linear function z subject to certain conditions determined by a set of linear inequalities. The variables are non-negative. There are also some other problems where we have to minimise a linear function (as an example, expenditure) subject to certain conditions determined by a set of linear inequalities with non-negative variables. Such problems are called **Linear Programming Problems.**

Before we proceed further, we now formally define some terms (which have been used above) which we shall be using in the linear programming problems :

The general structure of linear programming model consists of three basic components :

(1) Decision Variables : We need to evaluate various alternatives for arriving at the optimal value of the objective function. The variables in a linear program are a set of quantities that need to be determined in order to solve a problem. i.e., problem is solved when the best values of the variables have been identified. These variables are called decision variables. They are usually denoted by x, y (if there are two variables) or $x_1, x_2, \ldots, x_n$ if there are more variables.

In the example discussed above x, y are decision variables.

(2) The objective function : The objective function of each linear programming problem is expressed in terms of decision variables to optimize the criterion of optimality such as profit, cost, etc. It is expressed as :

Optimize (maximize or minimize)

$z = c_1x + c_2y$ or

$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$. In this chapter, we shall find the optimal value of the given objective function by the graphical method.

(3) The constraints : There are always certain limitations on the use of resources, e.g. labour, raw material, space, money, time etc. such limitations are being expressed as linear equalities or inequalities in terms of decision variables. The solution of a linear programming model must satisfy these constraints.

Now on we will denote a linear programming problem as an LP problem.

Thus, the general mathematical model of LP problem is as follows :

Find the values of decision variables x, y so as to optimize (maximize or minimize).

$$z = c_1x + c_2y$$

subject to the linear constraints,

$$a_{11}x + a_{12}y \quad (\leq, =, \geq)\ b_1$$
$$a_{21}x + a_{22}y \quad (\leq, =, \geq)\ b_2$$
$$a_{31}x + a_{32}y \quad (\leq, =, \geq)\ b_3$$
$$x \geq 0,\ y \geq 0$$

In general, we can write as the following :

Find the values of decision variables $x_1, x_2, \ldots, x_n$, so as to optimize (maximise or minimise)

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to the linear constrains,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad (\leq, =, \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad (\leq, =, \geq)\ b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad (\leq, =, \geq)\ b_m$$

and $x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0$

Here, **a_{ij}'s are coefficients representing the per unit contribution of decision variable x_j, to the value of objective function. a_{ij}'s are called the input-output coefficients and represent the amount of resource. a_{ij}'s can be positive, negative or zero. The b_i's represent the total availability of the ith resource.**

Let us take an example of LP model formulation.

Example 1 : A furniture firm manufactures chairs and tables. Each requires the use of three machines A, B or C. Production of one chair requires 2 hours on machine A, 1 hour on machine B and 1 hour on machine C. Each table requires 1 hour each on machines A and B and 3 hours on machine C. The profit realized by selling one chair is ₹ 300 while that from sale of a table is ₹ 600. The total time available per week on machine A is 70 hours, the time available on machine B is 40 hours and that on machine C is 90 hours. How many chairs and tables should be made per week so as to maximize profit ? Develop a mathematical formulation.

Solution : Let us represent the given data in a tabular form as following :

Machine	Chair number of hours	Table number of hours	Available time per week (in hours)
A	2	1	70
B	1	1	40
C	1	3	90
Profit per unit	₹ 300	₹ 600	

Let the number of chairs and tables manufactured respectively x and y.

Let z denote the total profit. Then $z = 300x + 600y$ **(i)**

It is given that a chair requires 2 hours on machine A and a table requires 1 hour on machine A. Therefore, the total time taken by machine A to produce x chairs and y tables is $(2x + y)$ hours.

This must be less than or equal to total hours available on machine A.

$\therefore \quad 2x + y \leq 70$ **(ii)**

It is given that a chair requires 1 hour on machine B and a table requires 1 hour on machine B. Therefore, total time taken by machine B to produce x chairs and y tables is $(x + y)$ hours. Total time available per week on machine B is 40 hours.

$\therefore \quad x + y \leq 40$ **(iii)**

Similarly, from the consideration of machine C we have the inequality

$x + 3y \leq 90$ **(Why ?) (iv)**

Since the number of chairs and tables cannot be negative.

$\therefore \quad x \geq 0$ and $y \geq 0$ **(v)**

Hence, the mathematical form of the given LPP is as follows :

Maximize $z = 300x + 600y$

Subject to $2x + y \leq 70$

$x + y \leq 40$

$x + 3y \leq 90$

and $x \geq 0, y \geq 0$.

We will now discuss how to find solutions to a linear programming problem. In this chapter we shall study only graphical method.

8.3 Graphical Method of Solving Linear Programming Problems

In this section first we shall discuss some definitions related to the solution of a linear programming problems.

Definition : The set of values of decision variables x_i ($i = 1, 2, ..., n$) which satisfy the constraints of an LP problem is said to constitute solution to that LP problem.

As an example,

Consider the LP problem.

Maximize $z = 300x + 600y$

subject to $2x + y \le 70$

$x + y \le 40$

$x + 3y \le 90$

and $x \ge 0, y \ge 0$

Here, $x = 1, y = 3$; $x = 7, y = 6$; $x = 10, y = 18$ etc. are solutions of this LP problem as they satisfy the constraints $2x + y \le 70$, $x + y \le 40$ and $x + 3y \le 90$ and $x \ge 0, y \ge 0$. Note that $x = 10$, $y = 30$ is not a solution because it does not safisfy $x + 3y \le 90$.

Feasible Solution : A set of values of the decision variables $x_1, x_2, ..., x_n$ is called a feasible solution of an LP problem, if it satisfies both the constraints and non-negativity conditions.

Infeasible Solution : An infeasible solution is a solution for which at least one constraint is violated.

Optimal feasible Solution : A feasible solution of an LP problem is said to be an optimal feasible solution, if it optimizes (maximizes or minimizes) the objective function.

Feasible region (solution region) : When we graph all the constraints, the feasible region is the set of all points which satisfy all the constraints including non-negativity constraints.

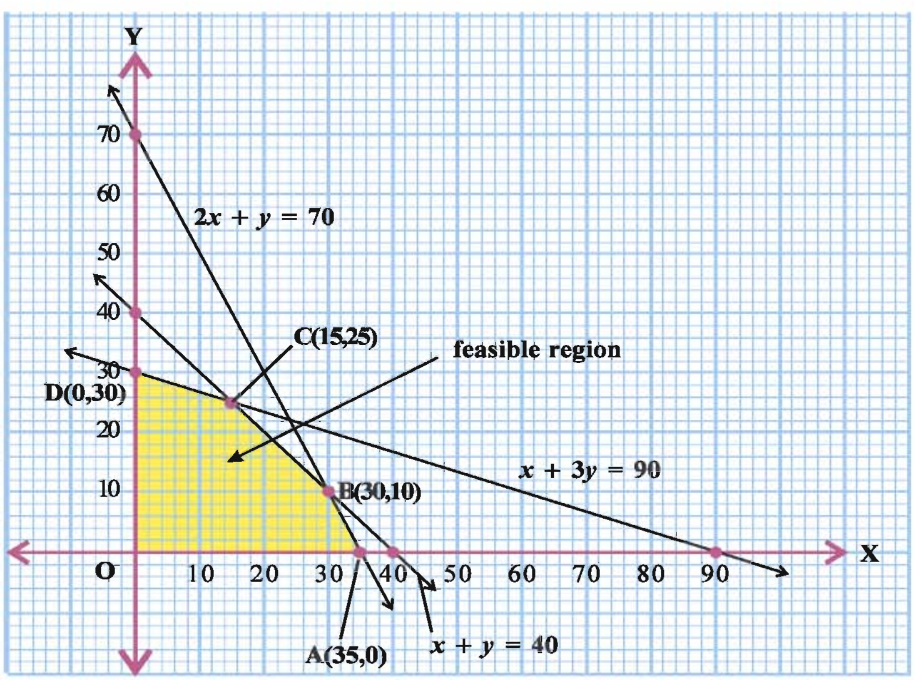

Figure 8.1

In figure 8.1, the region OABCD (yellow coloured) is the feasible region of Example 1.

The region other than the feasible region is called the **infeasible region.**

Note that points within and on the boundary of the feasible region represent feasible solutions of the constraints. In fig. 8.1, every point within or on the boundary of the feasible region OABCD represents feasible solution to the problem.

For example, the point (35, 0), (30, 10), (15, 25), (0, 30), (20, 0), (0, 10), (20, 10) etc. are some of the feasible solutions. The point (30, 20) is an infeasible solution of the problem. We see that every point in the feasible region OABCD satisfies all the constraints of example 1. We also observe that there are infinitely many points in the feasible region. Among them we have to find out one point which gives a maximum value of the objective function $z = 300x + 600y$. To handle this situation, we use the following theorems which are fundamental in solving linear programming problems. We shall not prove these theorems, we just state them.

Theorem 8.1 : Let R be the feasible region (convex polygon) for a linear programming problem and let $z = ax + by$ be the objective function. When z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 8.2 : Let R be the feasible region for a linear programming problem and let $z = ax + by$ be the objective function. If R is bounded, then the objective function z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

In the above example, the corner points (vertices) of the bounded (feasible) region are : O, A, B, C, D and their coordinates are (0, 0), (35, 0), (30, 10), (15, 25) and (0, 30) respectively. Let us now compute the values of z at these points. We have $z = 300x + 600y$.

Vertex of the feasible region	Corresponding value of z (in ₹)	
O(0, 0)	0	
A(35, 0)	10,500	
B(30, 10)	15,000	
C(15, 25)	19,500	←Maximum
D(0, 30)	18,000	

We observe that the maximum profit is earned by the firm by producing 15 chairs and 25 tables.

Note : If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R. (by theorem 8.1)

This method of solving linear programming problem is known as **Corner Point Method.**

Following steps can be used to solve an LP problem in two variables graphically by using corner-point method.

(1) Formulate the given LP problem in mathematical form, if it is not given in mathematical form.

(2) Find the feasible region of LP problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at the points.

(3) Evaluate the objective function $z = ax + by$ at each corner point. Let M and m respectively denote the largest and the smallest values of z at these points.

(4) When the feasible region is bounded, M and m are the maximum and minimum values of z.

(5) In case, the feasible region is unbounded, we have.

(i) M is the maximum value of z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, z has no maximum value.

(ii) m is the minimum value of z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, z has no minimum value.

We will now illustrate these steps of coner point method in some examples :

Example 2 : Solve the following linear programming problem graphically :

Maximize $z = 20x + 15y$

subject to $180x + 120y \leq 1500$

$x + y \leq 10$

and $x \geq 0,\ y \geq 0$

Solution : Since $x \geq 0$ and $y \geq 0$, the solution region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$,

Figure 8.2

(i) $180x + 120y \leq 1500$

$3x + 2y \leq 25$

Draw the line $3x + 2y = 25$

$$y = \frac{25 - 3x}{2}$$

x	0	5	$\frac{25}{3}$	1
y	$\frac{25}{2}$	5	0	11

Determine the region represented by $3x + 2y \leq 25$.

(ii) $x + y \leq 10$

Draw the line $x + y = 10$

$\therefore\ y = 10 - x$

x	0	10
y	10	0

Determine the region represented $x + y \leq 10$. Colour the intersection of the two regions. Also $x \geq 0$, $y \geq 0$. The yellow coloured region OABC in figure 8.2 is the feasible region. B(5, 5) is the point of intersection of $3x + 2y = 25$ and $x + y = 10$.

The corner points of OABC are O(0, 0), $A\left(\frac{25}{3}, 0\right)$, B(5, 5) and C(0, 10).

Vertex of the feasible region	Corresponding value of $z = 20x + 15y$
O(0, 0)	0
$A\left(\frac{25}{3}, 0\right)$	166.67
B(5, 5)	175
C(0, 10)	150

z is maximum at $x = 5$ and $y = 5$. Maximum value of $z = 175$.

Example 3 : Find the maximum and minimum value of $z = 2x + 5y$, subject to $3x + 2y \leq 6$, $-2x + 4y \leq 8$, $x + y \geq 1$, $x \geq 0$, $y \geq 0$ using corner point method.

Solution : Since $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$.

(1) $3x + 2y \leq 6$

Draw the line $3x + 2y = 6$

$$y = \frac{6 - 3x}{2}$$

x	0	2
y	3	0

Determine the region represented by $3x + 2y \leq 6$.

(2) $-2x + 4y \leq 8$

$\therefore\ -x + 2y \leq 4$

Draw the line $-x + 2y = 4$.

$\therefore\ y = \frac{x + 4}{2}$

x	0	2
y	2	3

Determine the region represented by $-x + 2y \leq 4$.

(3) $x + y \geq 1$

Draw the line $x + y = 1$ and determine the region represented by $x + y \geq 1$.

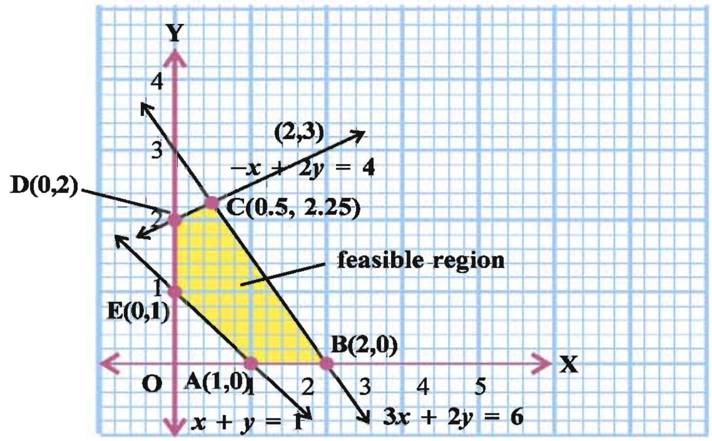

Figure 8.3

Colour the intersection of the three regions.

The yellow coloured region ABCDE in figure 8.3 is the feasible region. The point C(0.5, 2.25) is the point of intersection of $3x + 2y = 6$ and $-2x + 4y = 8$.

The corner points of ABCDE are A(1, 0), B(2, 0), C(0.5, 2.25), D(0, 2), E(0, 1).

Corner point	Value of $z = 2x + 5y$	
A(1, 0)	2	←Minimum
B(2, 0)	4	
C(0.5, 2.25)	12.25	←Maximum
D(0, 2)	10	
E(0, 1)	5	

Hence, $x = 1$, $y = 0$ minimizes $z = 2x + 5y$ and the minimum value is 2.

$x = 0.5$, $y = 2.25$ maximizes $z = 2x + 5y$ and the maximum value is 12.25.

Example 4 : Minimize $2x + 4y$ subject to $x + 2y \geq 10$; $3x + y \geq 10$; $x \geq 0$; $y \geq 0$.

Solution : Since $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$.

(1) $x + 2y \geq 10$

Draw the line $x + 2y = 10$

$\therefore \quad y = \frac{10 - x}{2}$

x	0	10
y	5	0

Determine the region represented by $x + 2y \geq 10$.

(2) $3x + y \geq 10$

Draw the line $3x + y = 10$.

$\therefore \quad y = 10 - 3x$

x	0	2
y	10	4

Determine the region represented by $3x + y \geq 10$.

Colour the intersection of the three regions. The feasible region is as shown in the figure 8.4 Observe that the feasible region is unbounded.

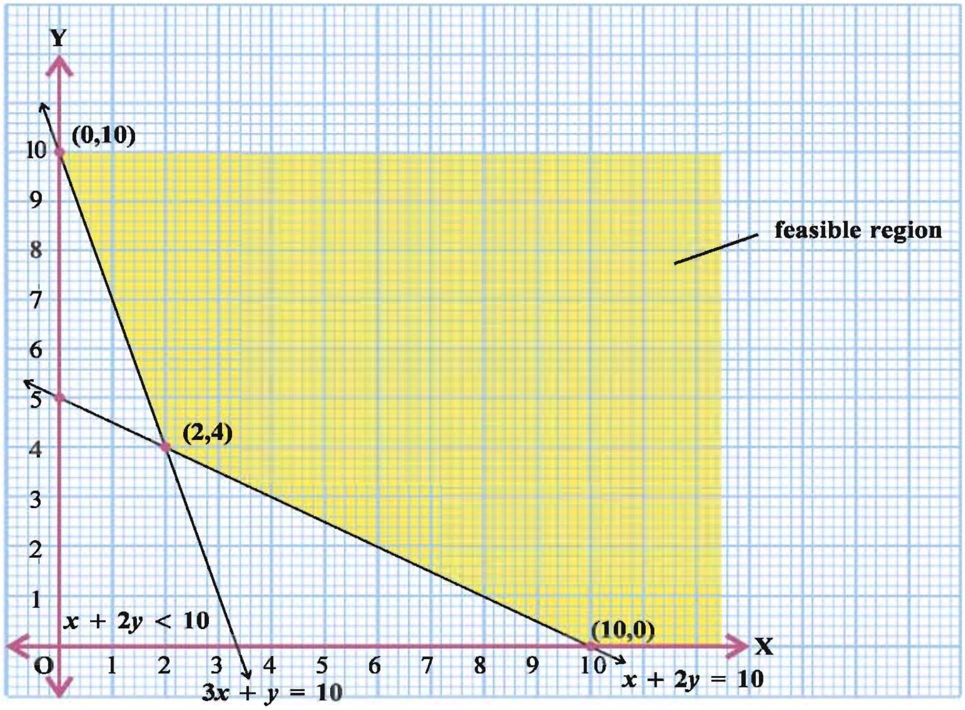

Figure 8.4

The corner points are (0, 10), (2, 4), (10, 0).

Corner point	Value of $z = 2x + 4y$
(0, 10)	40
(2, 4)	20
(10, 0)	20

From the table, we find that 20 may be the smallest value of z at the corner point (2, 4), (10, 0). Since the feasible region is unbounded, 20 may or may not be the minimum value of z. To decide this, we graph the inequality $2x + 4y < 20$ (see step 5(ii) of corner point method).

Now, $2x + 4y < 20$

$\therefore \quad x + 2y < 10$

We have to check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then 20 will not be the minimum value of z. Otherwise, 20 will be the minimum value of z. As shown in the figure 8.4, it has no common point with the feasible region. Hence, 20 is the minimum value of z. In fact, all the points on the line $x + 2y = 10$ give the same minimum value 20. Thus, there is an infinite number of points minimizing $z = 2x + 4y$ subject to the given constraints.

Example 5 : Determine graphically the minimum value of the objective function $z = -50x + 20y$ subject to the constraints.

$2x - y \geq -5$

$3x + y \geq 3$

$2x - 3y \leq 12$

$x \geq 0,\ y \geq 0,$

Solution : Since $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant and along $\overrightarrow{OX}$, $\overrightarrow{OY}$.

(1) $2x - y \geq -5$

Draw the line $2x - y = -5$

$\therefore \quad y = 2x + 5$

x	0	1
y	5	7

Determine the region represented by $2x - y \geq -5$.

(2) $3x + y \geq 3$

Draw the line $3x + y = 3$

x	0	1
y	3	0

Determine the region represented by $3x + y \geq 3$.

(3) $2x - 3y \leq 12$

Draw the line $2x - 3y = 12$

$\therefore \quad y = \frac{2x - 12}{3}$

x	9	6
y	2	0

Determine the region represented by $2x - 3y \leq 12$.

Colour the intersection of the three regions. The feasible region is as shown in the figure 8.5. Observe that the feasible region is unbounded.

The corner points are (0, 5), (0, 3), (1, 0) and (6, 0). We now evaluate z at the corner points.

Corner point	Value of $z = -50x + 20y$
A(0, 5)	100
B(0, 3)	60
C(1, 0)	–50
D(6, 0)	–300 ←Smallest

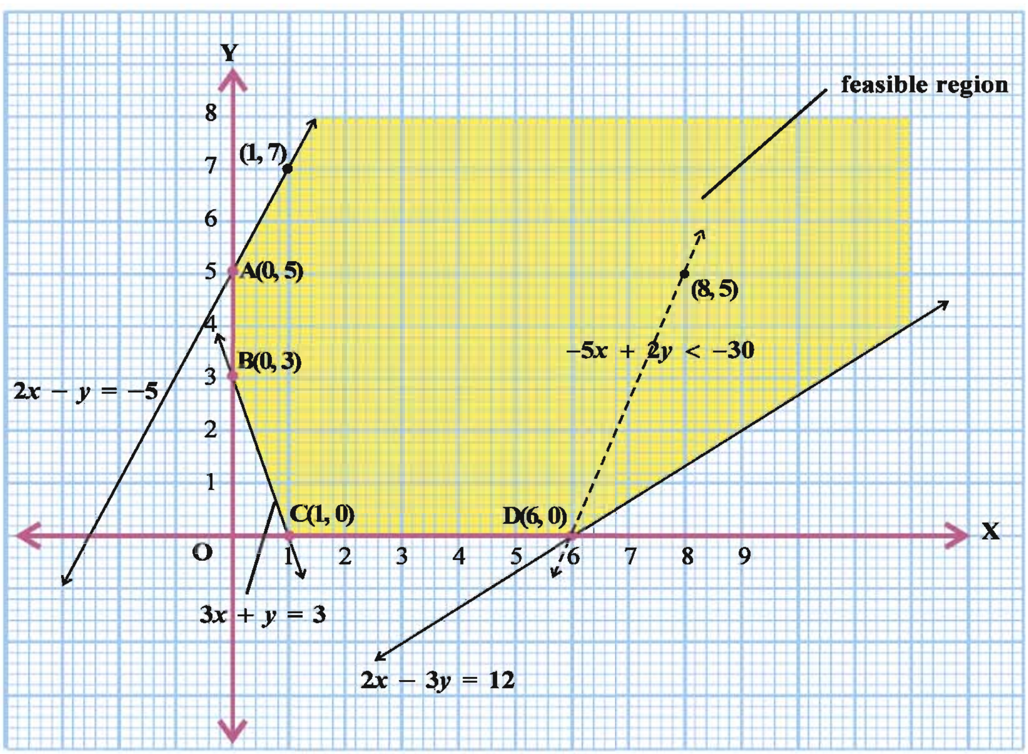

Figure 8.5

From the table, we find that -300 may be the smallest value of z at the corner point (6, 0). Since the feasible region is unbounded, -300 may or may not be the minimum value of z. To decide this, we graph the inequality $-50x + 20y < -300$ i.e. $-5x + 2y < -30$ and check whether the resulting open half plane has points in common with feasible region or not. If it has common points, then -300 will not be the minimum value of z. Otherwise -300 will be the minimum value of z. As shown in the figure 8.5, it has common points. Therefore, $z = -50x + 20y$ has no minimum value subject to the given constraints.

[In the above example, can you say whether $z = -50x + 20y$ has the maximum value 100 at (0, 5) ?]

Example 6 : Maximize $z = 3x + 4y$, if possible, subject to

$$x - y \leq -1$$
$$-x + y \leq 0$$
$$x \geq 0, y \geq 0$$

Solution : Let us graph the inequalities $x - y \leq -1$, $-x + y \leq 0$, $x \geq 0$ and $y \geq 0$.

From figure 8.6 we can see that there is no point satisfying all the constraints simultaneously. Thus, the problem has no feasible region and hence no feasible solution.

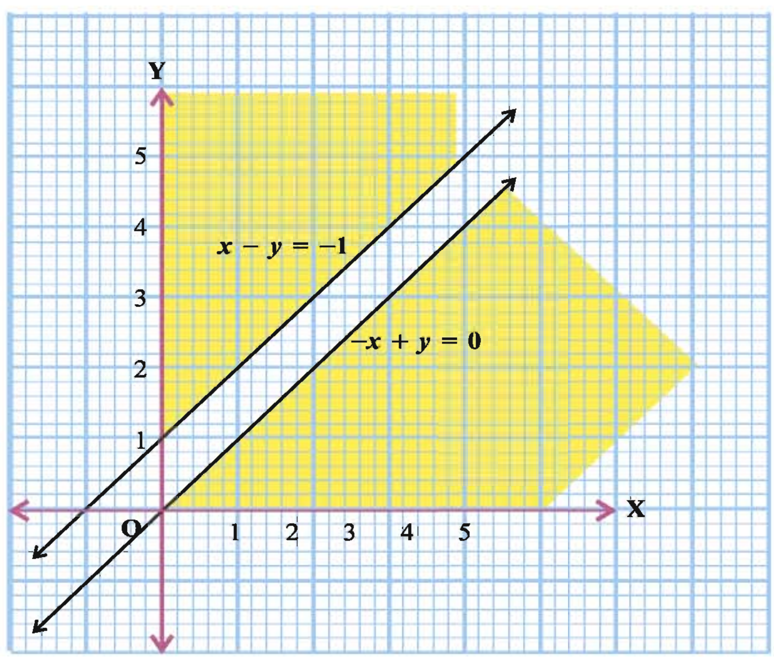

Figure 8.6

From the examples which we have discussed so far, we observed the following :

(1) The feasible region is always a convex region.

(2) The maximum (or minimum) solution of the objective function occurs at the corner of the feasible region. If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

Exercise 8.1

1. A company sells two different products A and B, making a profit of ₹ 40 and ₹ 30 per unit on them respectively. The products are produced in a common production process and are sold in two different markets. The production process has a total capacity of 3,000 man-hours. It takes three hours to produce a unit of type A and one hour to produce a unit of type B. The market has been surveyed and company officials feel that the maximum number of units of type A that can be sold is 8,000 and those of type B is 1200. Subject to these constraints, product can be sold in any combination. Formulate this problem as an LP problem mathematically to maximize the profit.
2. Vitamins A and B are found in foods F_1 and F_2. One unit of food F_1 contains three units of vitamin A and four units of Vitamin B. One unit of food F_2 contains six units of vitamin A and three units of vitamin B. One unit of food F_1 and F_2 costs ₹ 4 and ₹ 5 respectively. The minimum daily requirement (for a person) of vitamins A and B is 80 units and 100 units respectively. Assuming that anything in excess of the daily minimum requirement of A and B is not harmful, formulate this problem as an LP problem to find out the optimum mixture of foods F_1 and F_2 at the minimum cost which meets the daily minimum requirement of vitamins A and B.
3. A pension fund manager is considering investing in two shares A and B. It is estimated that,
   (1) share A will earn a dividend of 12 percent per annum and share B will earn 4 percent dividend per annum.
   (2) growths in the market value in one year of share A respectively are 10 paise per Re 1 invested and 20 paise per Re 1 invested in B.

   He requires to invest the maximum total sum which will give,
   (1) dividend income of at least ₹ 600 per annum; and
   (2) growth in one year of at least ₹ 1000 on the initial investment.

   Formulate this problem as an LP model to compute the minimum sum to be invested to meet the manager's objective.

Solve the following linear programming problems graphically (4 to 12) :

4. Maximize $z = 20x + 10y$

   subject to $x + 2y \leq 40$, $3x + y \geq 30$, $4x + 3y \geq 60$ and $x \geq 0$, $y \geq 0$
5. Maximize $z = 4x + y$

   subject to $x + y \leq 50$, $3x + y \leq 90$ and $x \geq 0$, $y \geq 0$
6. Minimize $z = 200x + 500y$

   subject to $x + 2y \geq 10$, $3x + 4y \leq 24$ and $x \geq 0$, $y \geq 0$
7. Minimize and maximize $z = 3x + 9y$

   subject to $x + 3y \leq 60$, $x + y \geq 10$, $x \geq y$, $x \geq 0$, $y \geq 0$
8. Minimize $z = 3x + 2y$

   subject to $x + y \geq 8$, $3x + 5y \leq 15$, $x \geq 0$, $y \geq 0$
9. Maximize $z = 3x + 4y$

   subject to $x + y \leq 4$, $x \geq 0$, $y \geq 0$
10. Maximize $z = 3x + 4y$

    subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$

11. Maximize $z = -x + 2y$

subject to $x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 6$

12. Minimize $z = 5x + 10y$

subject to $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x \geq 0$, $y \geq 0$

*

8.4 Different Types of Linear Programming Problems

Diet Problems : In this type of problems, we have to find the amount of different kinds of constituents / nutrients which should be included in a diet so as to minimize the cost of the desired diet and such that it contains a certain minimum amount of each constituent / nutrient.

Example 7 : A housewife wishes to mix together two kinds of food, X and Y, in such a way that the mixture contains at least 10 units of vitamim A, at least 12 units of vitamin B and at least 8 units of vitamin C. The vitamin contents of one *kg* of food is given below :

	Vitamin A	Vitamin B	Vitamin C
Food X	1	2	3
Food Y	2	2	1

One kg of food X costs ₹ 60 and one kg of food Y costs ₹ 100 . Find the least cost of the mixture which will produce the diet.

Solution : Let x *kg* of food X and y *kg* of food Y be mixed together to make the required diet.

1 *kg* of food X contains one unit of vitamin A and 1 *kg* of food Y contains 2 units of vitamin A.

Therefore, x *kg* of food X and y *kg* of food Y will contain $x + 2y$ units of vitamin A. It is given that the mixture should contain at least 10 units of vitamin A.

Therefore, $x + 2y \geq 10$ **(i)**

Similarly, x *kg* of food X and y *kg* of food Y will produce $2x + 2y$ units of vitamin B and $3x + y$ units of vitamin C. The minimum requirements of vitamin B and C are 12 and 8 units respectively.

$\therefore \quad 2x + 2y \geq 12$ **(ii)**

and $3x + y \geq 8$ **(iii)**

Since the quantity of food X and Y cannot be negative.

$\therefore \quad x \geq 0, y \geq 0$ **(iv)**

It is given that one *kg* of food X costs ₹ 60 and one *kg* of food Y costs ₹ 100. So, x *kg* of food X and y kg of food Y will cost ₹ $(60x + 100y)$. Thus, the given linear programming problem is

Minimize $z = 60x + 100y$

Subject to $x + 2y \geq 10$, $2x + 2y \geq 12$, $3x + y \geq 8$ and $x \geq 0$, $y \geq 0$.

Now let us solve this LP problem by graphical method.

To solve this LP problem, we draw the lines $x + 2y = 10$, $2x + 2y = 12$ i.e. $x + y = 6$ and $3x + y = 8$ and obtain the feasible region as shown in the figure 8.7, which is an unbounded one.

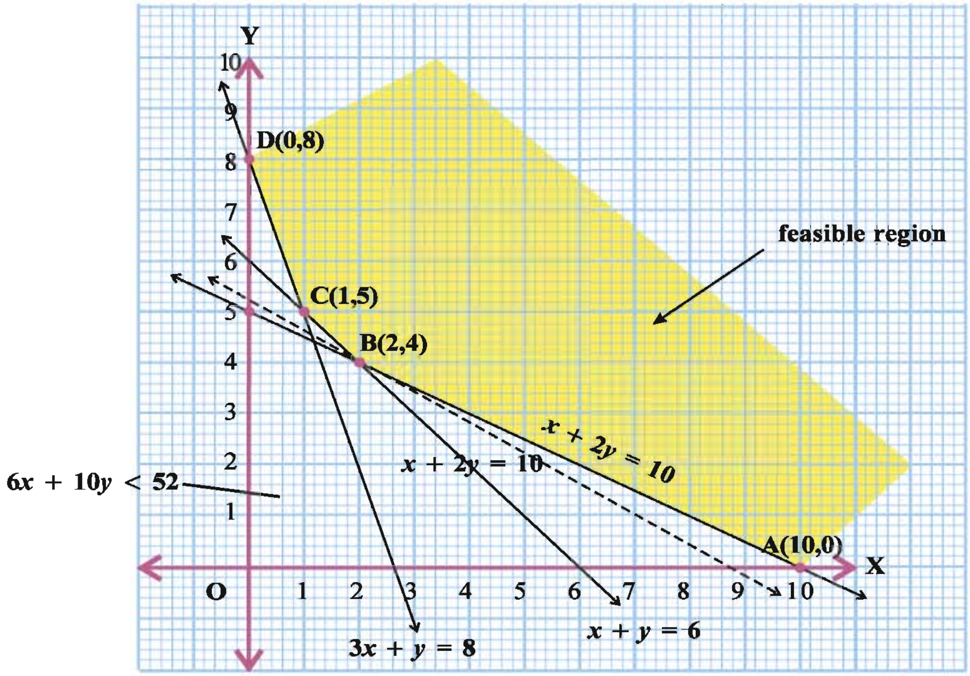

Figure 8.7

The corner points of the coloured region ABCD are A(10, 0), B(2, 4), C(1, 5) and D(0, 8). These points can also be obtained by solving simultaneously the equations of the corresponding intersecting lines,. The values of the objective function at these points are given in the following table :

Corner point	Value of the objective function $z = 60x + 100y$
A(10, 0)	600
B(2, 4)	520 ← Minimum
C(1, 5)	560
D(0, 8)	800

Clearly, z may be minimum at $x = 2$ and $y = 4$. Since the feasible region is unbounded, we have to graph the inequality $60x + 100y < 520$, i.e. $6x + 10y < 52$ and check whether the resulting open half plane has points in common with feasible region or not. We see from the figure 8.7 that it has no point common with the feasible region. So, z has minimum value equal to 520.

The minimum cost of the mixture is ₹ 520.

Manufacturing problems : In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed man-power, machine hours, labour hour per unit of product, warehouse space per unit of the output etc., in order to make maximum profit.

Example 8 : A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is atmost 24. It takes 1 hour to make a ring and 30 minutes to make a chain. The maximum number of hours available per day is 16. The profit on sell of a ring is ₹ 300 and that on sell of a chain is ₹ 190. Find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it an LP problem and solve it graphically.

Solution : Let the number of gold rings to be manufactured be x and that of chains be y. We construct the following table :

Item	Number	Time taken	Profit ₹
Gold ring	x	$1x$ hour	$300x$
Gold chain	y	$\frac{1}{2}y$ hour	$190y$
Total	$x + y$	$\left(x+\frac{1}{2}y\right)$ hour	$300x + 190y$

Our problem is to maximize the profit $z = 300x + 190y$ subject to constraints $x \geq 0,\ y \geq 0$ **(i)**

$x + \frac{1}{2}y \leq 16$

$\therefore\quad 2x + y \leq 32$ **(ii)**

and $x + y \leq 24$ **(iii)**

We draw the lines $2x + y = 32$ and $x + y = 24$ and obtain the feasible region as shown in the figure 8.8

Corner points of the feasible region OABC are O(0, 0), A(16, 0), B(8, 16), C(0, 24).

Let us evaluate z at these corner points.

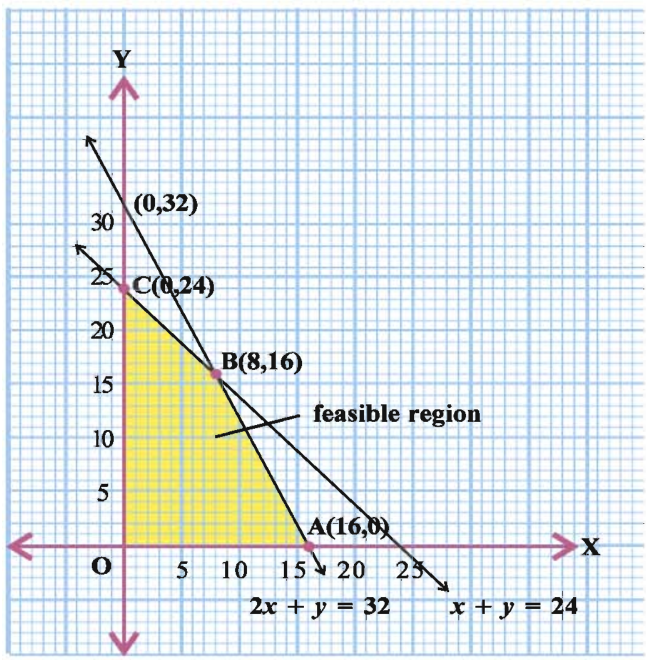

Figure 8.8

Corner point	Value of $z = 300x + 190y$ in ₹
(0, 0)	0
(16, 0)	4800
(8, 16)	5440 ← Maximum
(0, 24)	4560

We observe that profit is maximum when $x = 8$ and $y = 16$ and maximum profit is ₹ 5440.

Thus, to get maximum profit a firm has to produce 8 rings and 16 chains per day.

Transportation problems : In this type of problems, we have to determine transportation schedule for a commodity from different plants or factories situated at different relations to different markets in such a way that the total cost of transportation is minimum.

Example 9 : A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three construction companies P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost in ₹ of transporting 1000 bricks to the companies from the depots are given below :

From / To	P	Q	R
A	80	40	60
B	40	120	80

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum ?

Solution : The given information is as shown in the following figure.

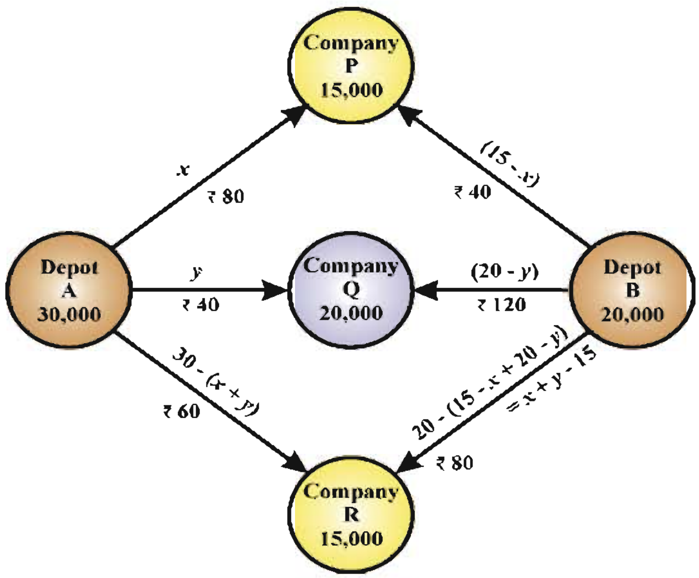

Figure 8.9

Let the depot A transport x thousand bricks to the company P and y thousand bricks to the company Q. Since the depot A has stock of 30,000 bricks, the remaining $30 - (x + y)$ thousand bricks will be transported to the company R. The number of bricks is always non-negative.

We have $x \geq 0$, $y \geq 0$ and $30 - (x + y) \geq 0$ i.e., $x + y \leq 30$ (i)

Now, the requirement of the company P is of 15,000 bricks and x thousand bricks are transported from the depot A, so the remaining $(15 - x)$ thousand bricks are to be transported from the depot B. The requirement of the company Q is of 20,000 bricks and y thousand bricks are transported from depot A. So the remaining $(20 - y)$ thousand bricks are to be transported from depot B. Now, depot B has $20 - (15 - x + 20 - y) = x + y - 15$ thousand bricks which are to be transported to the company R.

Also, $15 - x \geq 0$, $20 - y \geq 0$ and $x + y - 15 \geq 0$

$\therefore \quad x \leq 15$, $y \leq 20$ and $x + y \geq 15$ (ii)

The transportation cost from the depot A to the companies P, Q and R are respectively ₹ $80x$, ₹ $40y$ and ₹ $60(30 - (x + y))$. Similarly, the transportation cost from the depot B to the companies P, Q and R are respectively ₹ $40(15 - x)$, $120(20 - y)$ and $80(x + y - 15)$ respectively. Therefore, the total transportation cost z is given by

$z = 80x + 40y + 60(30 - x - y) + 40(15 - x) + 120(20 - y) + 80(x + y - 15)$

$\therefore \quad z = 60x - 60y + 3600$

Hence, the above LP problem can be stated mathematically as follows :

Minimize $z = 60x - 60y + 3600$

Subject to $x + y \leq 30$, $x \leq 15$, $y \leq 20$, $x + y \geq 15$ and $x \geq 0$, $y \geq 0$

Here, x and y are in thousands.

Let us solve this problem graphically. We draw the lines $x + y = 30$, $x = 15$, $y = 20$ and $x + y = 15$ and obtain the feasible region as shown in the figure 8.10.

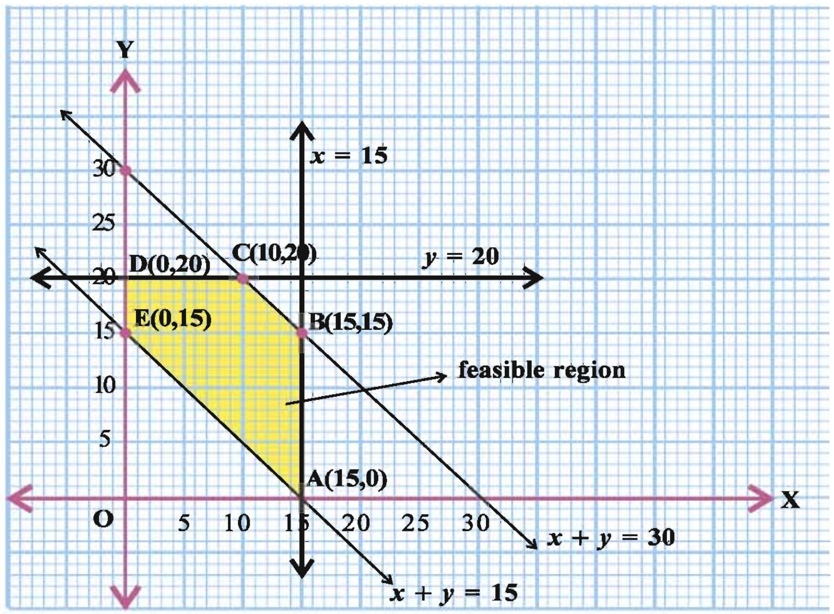

Figure 8.10

Corner points of the feasible region ABCDE are A(15, 0), B(15, 15), C(10, 20), D(0, 20), E(0, 15). Let us evaluate z at these corner points.

Corner point	Value of $z = 60x - 60y + 3600$
(15, 0)	4500
(15, 15)	3600
(10, 20)	3000
(0, 20)	2400 ← Minimum
(0, 15)	2700

Clearly, z is minimum at $x = 0$, $y = 20$ and the minimum value of z is 2400.

Thus, the manufacturer should supply 0, 20 and 10 thousand bricks to company P, Q and R from depot A and 15, 0 and 5 thousand bricks to company P, Q, R from depot B respectively.

In this case the minimum transportation cost will be ₹ 2400.

Marketing Problems : Linear programming can be used to determine the right mix of media exposure to use an advertising campaign. Suppose that the available media are radio, television and newspapers. The goal is to determine how many advertisements to place in each medium where the cost of placing an advertisement depends on the medium. Of course, we want to minimize the total cost of the advertising compaign and maximizing the mass where advertisement reaches.

Example 10 : An advertising agency wishes to reach two types of probable customers with annual income greater than one lakh rupees (target audience A) and customers with annual income less than one lakh rupees (target audience B). The total adverising budget is ₹. 2,00,000. One programme of TV advertising costs ₹ 50,000; one programme of radio advertising costs ₹ 20,000. For contract reasons, at least three programmes ought to be aired on TV and the number of radio programmes must be limited to 5. Surveys indicate that a single TV programme reaches 4,50,000 prospective customers in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 prospective customers in target audience A and 80,000 in target audience B. Determine the media mix to maximize the total reach.

Solution : Let us define the following decision variables :

Let x and y be the number of programmes to be aired on TV and radio respectively.

We are given that a single TV programme reaches 4,50,000 in target audience A and 50,000 in target audience B. One radio programme reaches 20,000 in target audience A and 80,000 in target audience B.

Hence, we have to maximize.

$$z = (4{,}50{,}000 + 50{,}000)x + (20{,}000 + 80{,}000)y$$

$$= 5{,}00{,}000x + 1{,}00{,}000y \qquad \text{(i)}$$

According to budget constraint we have

$$50{,}000x + 20{,}000y \le 2{,}00{,}000$$

$$\text{i.e., } 5x + 2y \le 20 \qquad \text{(ii)}$$

Also, there is number of programme constraints as at least 3 TV programmes and at the most 5 radio programme.

$$\therefore \quad x \ge 3 \text{ and } y \le 5 \qquad \text{(iii)}$$

Also, number of programmes is non-negative.

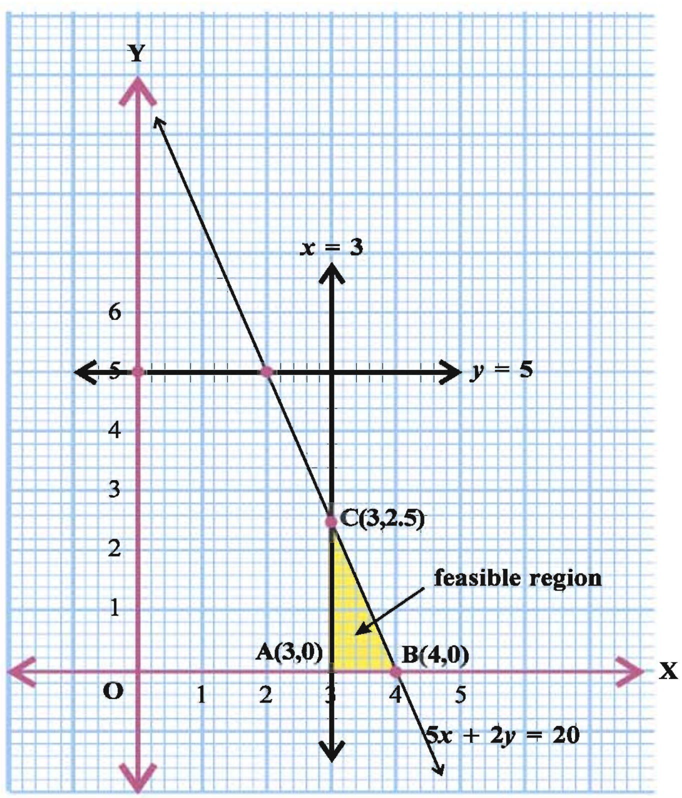

Figure 8.11

$\therefore \quad x \geq 0$ and $y \geq 0$ **(iv)**

Thus, LP problem is maximize $z = 5,00,000x + 1,00,000y$

subject to $5x + 2y \leq 20$, $x \geq 3$, $y \leq 5$ and $x \geq 0$, $y \geq 0$

Let us solve this problem graphically. We draw the lines $5x + 2y = 20$, $x = 3$, $y = 5$ and obtain the feasible region as shown in the figure 8.11.

Corner points of the feasible region ABC are A(3, 0), B(4, 0) and $C\left(3, \frac{5}{2}\right)$.

Let us evaluate z at thest corner points.

Corner point	Value of $z = 5,00,000x + 1,00,000y$
(3, 0)	15,00,000
(4, 0)	20,00,000 ⟵ Maximum
$\left(3, \frac{5}{2}\right)$	17,50,000

Since the maximum value of $z = 20,00,000$ occurs at the point B(4, 0), therefore, the agency must release 4 programmes on TV and no programme on radio to achieve the maximum target audiences.

Exercise 8

Use the graphical method to solve the following LP problems : (1 to 6)

1. Maximize $z = 2x + y$

   subject to $x + 2y \leq 10$, $x + y \leq 6$, $x - y \leq 2$, $x - 2y \leq 1$ and $x \geq 0$, $y \geq 0$

2. Minimize $z = -x + 2y$

   subject to $-x + 3y \leq 10$, $x + y \leq 6$, $x - y \leq 2$ and $x \geq 0$, $y \geq 0$

3. Minimize $z = 3x + 2y$

   subject to $5x + y \geq 10$, $x + y \geq 6$, $x + 4y \geq 12$ and $x \geq 0$, $y \geq 0$

4. Maximize $z = 7x + 3y$

   subject to $x + y \geq 3$, $x + y \leq 4$, $0 \leq x \leq \frac{5}{2}$, $0 \leq y \leq \frac{3}{2}$

5. Minimize $z = 20x + 10y$

   subject to $x + 2y \leq 40$, $3x + y \geq 30$, $4x + 3y \geq 60$ and $x \geq 0$, $y \geq 0$

6. Maximize $z = x + y$

   subject to $x + y \leq 1$, $-3x + y \geq 3$ and $x \geq 0$, $y \geq 0$

7. A factory owner purchases two types of machines A and B for his factory. The requirements and limitations for the machines are as folows :

Machine	Area occupied	Labour force	Daily out-put units
A	$1000\ m^2$	12 persons	60
B	$1200\ m^2$	8 persons	40

He has maximum area of 9000 m^2 available and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximize the daily output ? Formulate and solve the problem graphically.

8. A diet for a sick person must contain at least 4000 units of vitamin, 50 units of minerals and 1400 units of calories. Two foods A and B are available at a cost of ₹ 5 and ₹ 4 per unit respectively. One unit of food A contains 200 units of vitamins, 1 unit of minerals and 40 units of calories, while one unit of the food B contains 100 units of vitamins, 2 units of minerals and 40 units of calories. Find what combination of the foods A and B should be used to have minimum cost, but it must satisfy the requirements of the sick person. Formulate as an LP problem and solve it graphically.

9. A shopkeeper wishes to purchase a number of 5 *l* oil tins and 1 *kg* ghee tins. He has only ₹ 5760 to invest and has a space to store at most 20 items. A 5 *l* oil tin costs him ₹ 360 and a 1 *kg* ghee tin cost him ₹ 240. His expectation is that he can sell an oil tin at a profit of ₹ 22 and a ghee tin at a profit of ₹ 18. Assuming that he can sell all the items he can buy, how should he invest his money in order to maximize the profit ? Formulate this as a linear programming problem and solve it graphically.

10. One kind of cake requires 300 *g* of flour and 15 *g* of fat. Another kind of cake requires 150 *g* of flour and 30 *g* of fat. Find the maximum number of cakes which can be made from 7.5 *kg* of flour and 600 *g* of fat, assuming that there is no shortage of other ingredients used in making the cakes. Formulate it as an LP problem and solve it graphically.

11. An oil company has two depots A and B with capacities of 7000 *l* and 4000 *l* respectively. The company is to supply oil to three petrol pumps, D, E and F, whose requirements are 4500 *l*, 3000 *l* and 3500 *l* respectively. The distances (in *km*) between the depots and the petrol pumps is given in the following table : (Distance in *km*)

From \ To	A	B
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 *l* of oil is ₹ 1 per *km*. How should the delivery be scheduled in order that the transportation cost is minimum ? What is the minimum cost ?

12. An aeroplane can carry a maximum of 200 passangers. A profit of ₹ 1000 occurs on each executive class ticket and a profit of ₹ 600 occurs on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passangers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximize the profit for the airline. What is the maximum profit ?

13. A manufacturer produces two different models : X and Y, of the same product. Model X generates profit of ₹ 50 per unit and model Y generates profit of ₹ 30 per unit. Raw materials r_1 and r_2 are required for production. At least 18 *kg* of r_1 and 12 *kg* of r_2 must be used daily. Also at most 34 hours of labour are to be utilized. A quantity of 2 *kg* of r_1 is needed for model X and 1 *kg* of r_1 for model Y. For each of X and Y, 1 *kg* of r_2 is required. It takes 3 hours to manufacture model X and 2 hours to manufacture model Y. How many units of each model should be produced to maximize the profit ?

14. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

Section A (1 mark)

(1) Objective function of an LP problems is ☐

(a) a constant (b) a function to be optimized

(c) an inequality (d) a quadratic equation

(2) Let x and y be optimal solution of an LP problem, then ☐

(a) $z = \lambda x + (1 - \lambda)y$, $\lambda \in$ R is also an optimal solution

(b) $z = \lambda x + (1 - \lambda)y$, $0 \le \lambda \le 1$ gives an optimal solution.

(c) $z = \lambda x + (1 + \lambda)y$, $0 \le \lambda \le 1$ gives an optimal solution.

(d) $z = \lambda x + (1 + \lambda)y$, $\lambda \in$ R gives an optimal solution.

(3) The optimal value of the objective function is attained at the points ☐

(a) given by intersection of lines representing inequations with axes only

(b) given by intersection of lines representing inequations with X-axis only

(c) given by corner points of the feasible region

(d) at the origin

(4) The corner points of the feasible region determined by the system of linear constraints are (0, 10), (5, 5), (15, 15), (0, 20). Let $z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of z occurs at both the pooints (15, 15) and (0, 20) is ...... ☐

(a) $p = q$ (b) $p = 2q$ (c) $q = 2p$ (d) $q = 3p$

(5) Which of the following statements is correct ? ☐

(a) Every LP problem has at least one optimal solution.

(b) Every LP problem has a unique optimal solution.

(c) If an LP problem has two optimal solutions, then it has infinitely many solutions.

(d) If a feasible region is unbounded then LP problem has no solution.

(6) In solving the LP problem : ☐

"Minimize $z = 6x + 10y$

subject to $x \ge 6$, $y \ge 2$, $2x + y \ge 10$, $x \ge 0$, $y \ge 0$." redundant constraints are

(a) $x \ge 6$, $y \ge 2$ (b) $2x + y \ge 10$, $x \ge 0$, $y \ge 0$

(c) $x \ge 6$ (d) $x \ge 6$, $y \ge 0$

(7) A feasible solution to an LP problem, ☐

(a) must satisfy all of the problem's constraints simultaneously

(b) need not satisfy all of the constraints, only some of them.

(c) must be a corner point of the feasible region.

(d) must optimize the value of the objective function.

Section B (2 marks)

(8) For the LP problem ☐

"Maximize $z = x + 4y$

subject to $3x + 6y \le 6$, $4x + 8y \ge 16$ and $x \ge 0$, $y \ge 0$."

(a) 4 (b) 8

(c) feasible region is unbounded (d) has no feasible region

(9) For the LP problem

Maximize $z = 2x + 3y$

the coordinates of the corner points of the bounded feasible region are A(3, 3), B(20, 3), C(20, 10), D(18, 12) and E(12, 12). The maximum value of z is ......

(a) 72 (b) 80 (c) 82 (d) 70

(10) For the LP problem

Minimize $z = 2x + 3y$

the coordinates of the corner points of the bounded feasible region are A(3, 3), B(20, 3), C(20, 10), D(18, 12) and E(12, 12). The minimum value of z is ......

(a) 49 (b) 15 (c) 10 (d) 05

Section C (3 marks)

(11) Solution of the following LP problem

Maximize $z = 2x + 6y$

subject to $-x + y \le 1$, $2x + y \le 2$ and $x \ge 0$, $y \ge 0$" is

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$ (c) $\frac{26}{3}$ (d) no feasible region

(12) Solution of the following LP problem

Minimize $z = -3x + 2y$

subject to $0 \le x \le 4$, $1 \le y \le 6$, $x + y \le 5$ is

(a) -10 (b) 0 (c) 2 (d) 10

Section D (4 marks)

(13) The following graph represents a feasible region. Minimum value of $z = 5x + 4y$ is ......

(a) 150 (b) 145 (c) 160 (d) 250

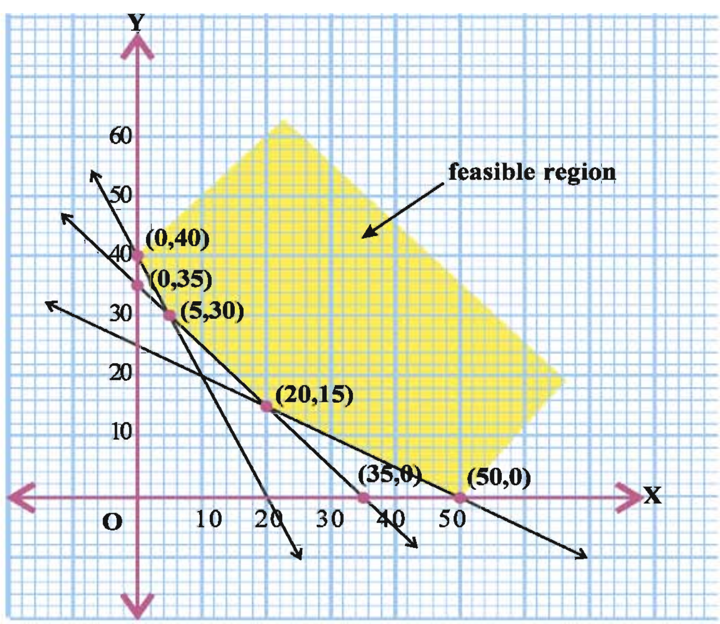

Figure 8.12

(14) Corner points of the bounded feasible region for an LP problem are (0, 4), (6, 0), (12, 0), (12, 16) and (0, 10). Let $z = 8x + 12y$ be the objective function. Match the following : ☐

(i) Minimum value of z occurs at ...... (ii) Maximum value of z occurs at ......

(iii) Maximum of z is ...... (iv) Minimum of z is ......

(a) (i) (6, 0) (ii) (12, 0) (iii) 288 (iv) 48

(b) (i) (0, 4) (ii) (12, 16) (iii) 288 (iv) 48

(c) (i) (0, 4) (ii) (12, 16) (iii) 288 (iv) 96

(c) (i) (6, 0) (ii) (12, 0) (iii) 288 (iv) 96

*

Summary

We have studied the following points in this chapter :

1. Mathematical formulation of linear programming problems.
2. Meaning of the terms : Decision variables, the objective function, the constraints.
3. Graphical method of solving linear programming problems
4. Meaning of the terms : feasible solution, infeasible solution, optimal feasible solution, feasible region, infeasible region.

Fields of Indian mathematics

Some of the areas of mathematics studied in ancient and medieval India include the following :

Arithmetic : Decimal system, Negative numbers (Brahmagupta), Zero (Hindu numeral system), Binary numeral system, the modern positional notation numeral system, Floating point numbers (Kerala school of astronomy and mathematics), Number theory, Infinity (Yajur Veda), Transfinite numbers

Geometry : Square roots (Bakhshali approximation), Cube roots (Mahavira), Pythagorean triples (Sulba Sutras; Baudhayana and Apastamba) statement of the Pythagorean theorem without proof), Transformation (Panini), Pascal's triangle (Pingala)

Algebra : Quadratic equations (Sulba Sutras, Aryabhata, and Brahmagupta), Cubic equations and Quartic equations (biquadratic equations) (Mahavira and Bhaskara II)

Mathematical logic : Formal grammars, formal language theory, the Panini–Backus form (Panini), Recursion (Panini)

General mathematics : Fibonacci numbers (Pingala), Earliest forms of Morse code (Pingala), infinite series, Logarithms, indices (Jain mathematics), Algorithms, Algorism (Aryabhata and Brahmagupta)

Trigonometry : Trigonometric functions (Surya Siddhanta and Aryabhata), Trigonometric series (Madhava and Kerala school)